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TWO RESULTS ON THE REARRANGEMENT OF SERIES

*Harry Miller** *and Ekrem Öztürk***

* *Mathematical Department, Faculty of Natural Sciences and Mathematics, University of Sarajevo Vojvode Putnika 43, Yugoslavia*

** *Erciyes University, Faculty of Arts and Sciences Department of Mathematics, 38039 Kayseri, Turkey*

ABSTRACT

In previous papers ([4] and [5]) E. Öztürk proved several results about the existence of rearrangements of series with certain properties. The purpose of this paper is to show that the rearrangements constructed in two of these theorems are rare, in the sense that the set of rearrangements having the desired property forms a set of the first Baire category in the set of all rearrangements. These results are related to an earlier theorem of this type by H. Miller [3].

1. PRELIMINARIES

We will follow the notation introduced in [3]. In this paper the word permutation will be used to denote any function P,

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$P:N\to N$, the natural numbers, whose domain and range is N and that is also one to one. The set of all permutations will be denoted by the simbol $\mathcal{P}$.

Definition. *For each* P, Q *in* $\mathcal{P}$, *define* $d(P,Q) = 1/n$ *if* $P(i) \neq Q(i)$ *for each* $i=1,2,\dots,n-1$ *and* $P(n) = Q(n)$. *Furthermore, define* $d(P,P) = 0$ *for each* $P\in\mathcal{P}$.

As shown in [3] $(\mathcal{P},d)$ is a metric space that is incomplete; but is of the second Baire category, i.e. $\mathcal{P}$ cannot be expressed as a countable union of nowhere dense subsets of $(\mathcal{P},d)$.

The following theorem, discovered by Riemann in 1849, can be found in most standard text books on Advanced Calculus, for example see [1], page 368.

Theorem R. *Suppose* $\sum_{n=1}^{\infty} a_n$ *is a series of real numbers satisfying:* $\lim_{n\to\infty} a_n=0$ *and* $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} v_n = +\infty$, *where* $u_n=\max(a_n, 0)$ *and* $v_n = \max(-a_n, 0)$ *for each* n *in* N. *Suppose further that* x *and* y *are numbers in the closed interval* $[-\infty, +\infty]$, *with* $x \leq y$.

Then there exists P *in* $\mathcal{P}$ *such that:*

$$\liminf_{n\to\infty} \sum_{i=1}^{n} a_{P(i)} = x \text{ and } \limsup_{n\to\infty} \sum_{i=1}^{n} a_{P(i)} = y.$$

Suppose that the series $\sum_{n=1}^{\infty} a_n$ satisfies the hypotheses of Theorem R. Let $\mathcal{P}_r$ denote the set of all P in $\mathcal{P}$ such that: $\sum_{n=1}^{\infty} a_{P(n)} = r$. The following theorem appears in [3].

Theorem M. *The set* $\cup[\mathcal{P}_r$: r *in* $(-\infty, +\infty)]$ *is of the first Baire category in* $(\mathcal{P}, d)$.

The following theorem appears in [4].

Theorem $\ddot{O}_1$. *Suppose that the series* $\sum_{i=1}^{\infty} a_i x_i$ *is conditionally convergent and that* $\lim_{n\to\infty} a_{P(n)} x_n = 0$ *for each* $P\in\mathcal{P}$.

Moreover assume that the sequence (a_n) *has infinitely many positive terms and infinitely many negative terms. Then, for any* r *in* $\mathbb{R}$ *(the reals) there exists a permutation* P *in* $\mathcal{P}$ *such that* $\sum_{i=1}^{\infty} a_{P(i)} x_i = r$.

Notice that Theorem $\ddot{O}_1$ extends Theorem R.

Consider $A = (a_{nk})$; $n, k = 1,2,\ldots$, an infinite matrix whose entrees are real numbers. For each sequence of reals (x_k) one can consider the transformed sequence (y_n), where $y_n = \sum_{k=1}^{\infty} a_{nk} x_k$ for each n. The matrix is called regular if $\lim_{n\to\infty} x_n = r$ implies $\lim_{n\to\infty} y_n = r$, where (y_n) is the transform of the sequence (x_n).

It is well known that A is regular if and only if the following three conditions are satisfied (see [2]).

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty,$$

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1, \quad \text{and}$$

$$\lim_{n\to\infty} a_{nk} = 0, \quad \text{for each fixed } k.$$

A is said to be strongly regular if A is regular and $\lim_{n\to\infty} \sum_{k=1}^{\infty} |a_{nk} - a_{n,k+1}| = 0$.

Let $z'_n = \sum_{k=1}^{\infty} a_{nk} x_k$, $z''_n = \sum_{k=1}^{\infty} b_{nk} x_k$, where $A = (a_{nk})$ and $B = (b_{nk})$ are regular matrices and (x_k) is a sequence of real numbers. The matrices A and B are said to be absolutely equivalent for a given class of sequences (x_k) if

$$z'_n - z''_n \to 0 \quad \text{as} \quad n\to\infty$$

for each sequence (x_k) in the class; i.e., either z'_n , z''_n

both tend to the same limit, or else neither of them tends to a limit, but their differences tend to zero (for detailed information see [2]).

Definition. *If* $A = (a_{nk})$ *is an infinite matrix and* $p \in P$, *then* A_p *is the infinite matrix defined by* $A_p = (a_{n,p(k)})$.

Definition. *A permutation* $p \in P$ *is said to move infinitely many integers if the set* $(n \in N: p(n) \neq n)$ is infinite.

The following theorem is an immediate corollary of Theorem 2.1 in [5].

Theorem $\ddot{O}_2$. *Suppose* $A = (a_{nk})$ *is a strongly regular matrix. There exists a* $p \in P$ *that moves infinitely many integers such that* A *and* A_p *are absolutely equivalent for the class* l_∞ *of all bounded real sequences.*

The purpose of this paper is to show that the sets of permutations satisfying the conditions of Theorems O_1 and O_2 are both sets of the first Baire category in P.

2. RESULTS

Our first results shows that the set of permutations satisfying the conditions in Theorem $\ddot{O}_1$ forms a set of the first Baire category in P.

Theorem 1. *Suppose that the series* $\sum_{k=1}^{\infty} a_k x_k$ *is conditionally convergent and that the sequence* (a_k) *has infinitely many positive terms and infinitely many negative terms. Then* $\{p \in P: \sum_{k=1}^{\infty} a_{p(k)} x_k$ *is convergent}* *is a set of the first Baire category.*

Proof. Let $T_n = \{P\in P: |\sum_{k=n}^{m} a_{P(k)}x_k|<1$ for all m, $m \geq n\}$. Clearly $\bigcup_{n=1}^{\infty} T_n \supseteq \{P\in P: \sum_{k=1}^{\infty} a_{P(k)}x_k$ is convergent$\}$. We will now show that each set T_n is nowhere dense in P.

Take any permutation P_1 and $r>0$. There exists an $n_1>n$, such that $P(k) = P_1(k)$ for every $k=1,2,\ldots,n_1$ implies $P\in K(P_1,r)$, where $K(P_1, r)$ is the open ball in (P,d) with center at P_1 and radius r. The fact that $\sum_{k=1}^{\infty} a_k x_k$ is a conditionally convergent series implies that there exists a subsequence (m_k) of the positive integers such that $a_{m_k}x_{m_k}>0$ for all k and $\sum_{k=1}^{\infty} a_{m_k}x_{m_k} = \infty$. We may assume that $m_1>P_1(k)$ for each $k=1,2,\ldots,n_1$.

There exists a q such that

$$\sum_{k=1}^{n_1} a_{P_1(k)}x_k + \sum_{k=1}^{q} a_{m_k}x_{m_k} > 1.$$

Now, let P_2 be any permutation such that:

$P_2(k) = P_1(k)$ for every $k=1,\ldots,n_1$;

$P_2(m_k) = m_k$ for every $k=1,\ldots,q$ and

$a_{P_2(k)}x_k \geq 0$ for every k, $n_1<k\leq m_q$.

Such a permutation exists since (a_k) has infinitely many positive and infinitely many negative terms. Clearly $P_2\notin T_n$ and $P_2\in K(P_1,r)$. Furthermore, if P agrees with P_2, in the first m_q places, then $P\in K(P_1,r)$ and $P\notin T_n$. Therefore there exists an open ball K, $K\subseteq K(P_1,r)$ and $K\cap T_n = \emptyset$. Therefore T_n is nowhere dense in P.

We will now show that the set of permutations in Theorem Ö$_2$ forms a set of the first Baire category in P.

Theorem 2. *Suppose that* $A = (a_{nk})$ *is a regular matrix. Then* $S = \{P \in P: A$ *and* A *are absolutely equivalent with respect to* $l_\infty\}$ *is a set of the first Baire category in* P.

Proof. Let $x = (0, 1, 0, 1, 0, 1, \ldots)$ and let $S' = \{P \in P: \lim_{n\to\infty}((A_p x)_n - (Ax)_n) = 0\}$. Clearly $S' \supseteq S$. Set $T_n = \{P \in P: |(A_p x)_k - (Ax)_k| < 1/5$ for all $k \geq n\}$ and $T = \bigcup_{n=1}^{\infty} T_n$. Clearly $T \supseteq S' \supseteq S$. We will show that each T_n is nowhere dense in P.

To see this take any permutation p_1 in P and any $r>0$ and examine the open ball $K(P_1,r)$. There exists an odd integer n_1 such that $p(k) = p_1(k)$ for every $k=1,2,\ldots,n_1$ implies $P \in K(P_1,r)$. Since A_{P_1} is a regular matrix (this is an easy exercise), there exists an n_2, $n_2>n_1$ such that

$$|a_{n_2 p_1(i)}| < 1/10n_1 \quad \text{for every } i=1,2,\ldots,n_1 \text{ and}$$

$$\left|\sum_{i=1}^{\infty} a_{n_2 p_1(i)} - 1\right| < 1/10.$$

There exists $n_3>n_1$ such that

$$\sum_{i=n_1+1}^{n_3} a_{n_2 p_1(i)} > 8/10 \quad \text{and} \quad \sum_{i=n_3+1}^{\infty} |a_{n_2 p_1(i)}| < 1/10.$$

In the following arb will be used for the word arbitrary. There exist permutations P_2 and p_3 which both agree with P_1 in the first n_1 places (i.e. $p_1(i) = p_2(i) = p_3(i)$ for $i=1,\ldots,n_1$) and such that

i	even n_1+1	odd n_1+2	even n_1+3	odd n_1+4	even n_1+5	....	even n_1+2n_3-1	odd n_1+2n_3
$P_2(i)$	$P_1(n_1+1)$	arb	$P_1(n_1+2)$	arb	$P_1(n_1+3)$	....	$P_1(n_3)$	arb
$P_3(i)$	arb	$P_1(n_1+1)$	arb	$P_1(n_1+2)$	arb	....	arb	$P_1(n_3)$

We will now examine

$$(A_{P_2}(x))_{n_2} - (A_{P_3}(x))_{n_2}.$$

The first n_1 terms of these sums are the same and therefore this difference is equal to $\sum_{i=n_1+1}^{\infty} a_{n_2P_2(i)} \cdot x_i - \sum_{i=n_1+1}^{\infty} a_{n_2P_3(i)} \cdot x_i$. The first sum is equal to

$$\sum_{i=n_1+1}^{n_3} a_{n_2P_1(i)} + C_1, \text{ and}$$

$$|C_1| \leq \sum_{i=n_3+1}^{\infty} |a_{n_2P_1(i)}| < 1/10,$$

and therefore is greater than 7/10.

The absolute value of the second sum is clearly less than $\sum_{i=n_3+1}^{\infty} |a_{n_2P_1(i)}|$ and is hence less than 1/10.

Therefore $(A_{P_2}(x))_{n_2} - (A_{P_3}(x))_{n_2} > 6/10$. Hence either $|(A_{P_2}(x))_{n_2} - (A_{P_1}(x))_{n_2}| > 3/10 > 1/5$ or $|(A_{P_3}(x))_{n_2} - (A_{P_1}(x))_{n_2}| > 3/10 > 1/5$.

Call P^* the permutation (either P_2 or P_3) satisfying the above inequality. Now suppose that P is any permutation agreeing with P^* in the first $n_1+2n_3+n_4$ places. Then $P \in K(P_1, r)$ and if n_4 is sufficiently large $P \notin T_n$. Therefore there exists a ball K, K $\subseteq$ K(P_1,r) such that $K \cap T_n = \emptyset$; hence T_n is nowhere dense, completing the proof.

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REZIME

DVA REZULTATA O PREUREDJENJU REDOVA

U prethodnim radovima ([4], [5]) E. Öztürk je dokazao neke rezultate o postojanju preuredjenja redova sa nekim osobinama. Cilj ovog rada je da se pokaže da je preuredjenje konstruisano u ovim dvema teoremama retko, u smislu da skup preuredjenja koji ima očekivane osobine obrazuje skup prve Berove kategorije u skupu svih preuredjenja. Ovi rezultati su povezani sa ranijom teoremom ovog tipa od H. Millera [3].

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