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Serija za matematiku, 17,1(1987)

REVIEW OF RESEARCH
Faculty of Science
University of Novi Sad
Mathematics Series, 17,1(1987)

ON THE FORM OF THE APPROXIMATE SOLUTION OF A PARTIAL DIFFERENTIAL EQUATION

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ABSTRACT

In this paper a linear partial differential equation with constant coefficients is observed in the field of Mikusiński operators. A new form of the approximate solution is constructed and the error of approximation is estimated.

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We are going to observe the linear partial differential equation with constant coefficients:

$$(1)\qquad \sum_{\mu=0}^{m}\sum_{\nu=0}^{1}\alpha_{\mu,\nu}\frac{\partial^{\mu+\nu}x(\lambda,t)}{\partial\lambda^{\mu}\partial t^{\nu}} = 0 \qquad \lambda_1 \leqslant \lambda \leqslant \lambda_2$$

with conditions:

$$(2)\qquad \frac{\partial^{\mu}x(\lambda,0)}{\partial\lambda^{\mu}} = 0 \qquad \mu = 0,\dots,m$$

AMS Mathematics Subject Classification (1980): Primary 44A40, Secondary 65M15.

Key words and phrases: The field of Mikusiński's operators, partial differential equations

(3) $$\frac{\partial^{\mu} x(0,t)}{\partial\lambda^{\mu}} = 0, \quad t > 0, \ \mu = 0,\ldots,m-2 \geqslant 0,$$
$$\frac{\partial^{m-1} x(0,t)}{\partial\lambda^{m-1}} = 1, \qquad t > 0.$$

In the field of Mikusinski operators M, the equation

(4) $$\sum_{\mu=0}^{m} \sum_{\nu=0}^{1} \alpha_{\mu,\nu} s^{\nu} x^{(\mu)}(\lambda) = 0$$

with conditions:

(5) $$x(0) = 0, \ x'(0) = 0, \ldots, \ x^{(m-1)}(0) = \ell$$

corresponds to equation (1) with conditions (2) and (3).

The solution of equation (4) is of the form:

(6) $$x(\lambda) = \sum_{j=1}^{m} b_j exp(\lambda\omega_j), \qquad \omega_j = \sum_{i=0}^{\infty} c_{i,j}\cdot\ell^{\frac{i-p_j}{q_j}}$$

where ω_j are the solutions of the characteristic equation and b_j are the coefficients determined by (5).

The approximate solution of equation (4) on the interval $[0,T]$ has the form:

(7) $$\tilde{x}(\lambda) = \sum_{j=1}^{m} b_j exp(\lambda\tilde{\omega}_j), \qquad \tilde{\omega}_j = \sum_{i=0}^{i} c_{i,j}\cdot\ell^{\frac{i-p_j}{q_j}} .$$

As in [3], in this paper we divide the interval $[0,T]$ into n equal subintervals, $[0,T_1]$, $[T_2,T_3]$,..., $[T_{n-1},T]$. In [3] we constructed the approximate solution of equation (1) with conditions (2) and (3) on the interval $[T_{n-1},T]$ in several steps. The form of this approximate solution implied that we had to construct it on each subinterval, and it depended (out of the first one) on the approximate solution on the previous one. In this paper, we shall give a new form of the approximate solution on the last subinterval $[T_{n-1},T]$. Its construction is easier than the one in [3], since it requires only the approximate solution of the first subinterval. At the same time, it turns out that the error of approximation is much better than before.

The correct solution on the first subinterval $[0,T]$ has the form (6) and the approximate one has the form (7). On the subinterval $[T_1,T_2]$ the exact solution of equation (4) is of the form:

$$x_2(\lambda) = x_h(\lambda) + \frac{e^{-hs}}{Q}\int_0^{\lambda} F_1(\kappa)x_h(\lambda-\kappa)d\kappa \tag{8}$$

where $x_h(\lambda)$ is given by (6),

$$Q^{-1} = \frac{I}{\ell(\alpha_{m,0} + \alpha_{m,1}s)} \tag{9}$$

and

$$F_1(\lambda) = \sum_{\mu=0}^{m} \alpha_{\mu,1} \left.\frac{\partial^{\mu}x(\lambda,t)}{\partial\lambda^{\mu}}\right|_{t=T_1} \tag{10}$$

The approximate solution on the interval $[T_1,T_2]$ is:

$$\tilde{x}_2(\lambda) = \tilde{x}(\lambda) + \int_0^{\lambda} \tilde{F}_1(\kappa)\tilde{x}(\lambda-\kappa)d\kappa, \tag{11}$$

where $\tilde{x}(\lambda)$ is of the form (7), Q is of the form (9), while

$$\tilde{F}_1(\lambda) = \sum_{\mu=0}^{m} \alpha_{\mu,1} \left.\frac{\partial^{\mu}\tilde{x}(\lambda,t)}{\partial\lambda^{\mu}}\right|_{t=T_1}. \tag{12}$$

Repeating this procedure, one obtains the exact solution on the interval $[T_{n-1},T]$:

$$x_n(\lambda) = x(\lambda) + \frac{e^{-(n-1)hs}}{Q}\int_0^{\lambda} F_{n-1}(\kappa)x(\lambda-\kappa)d\kappa, \tag{13}$$

where $x(\lambda)$ is given by (6), Q is given by (9) and

$$F_{n-1}(\lambda) = \sum_{\mu=1}^{m} \alpha_{\mu,1} \left.\frac{\partial^{\mu}x_{n-1}(\lambda,t)}{\partial\lambda^{\mu}}\right|_{t=T_{n-1}}, \tag{14}$$

while the approximate solution on the interval $[T_{n-1},T]$ is

$$(15)\qquad \tilde{x}_n(\lambda) = \tilde{x}(\lambda) + \frac{e^{-(n-1)hs}}{Q}\int_0^{\lambda} \tilde{F}_{n-1}(\kappa)\tilde{x}(\lambda-\kappa)d\kappa,$$

where

$$(16)\qquad \tilde{F}_{n-1}(\kappa) = \sum_{\mu=1}^{m} \alpha_{\mu,1}\left.\frac{\partial^{\mu}\tilde{x}_{n-1}(\lambda,t)}{\partial\lambda^{\mu}}\right|_{t=T_1}.$$

In order to get a new form of the exact and the approximate solution let us prove:

Lemma 1. *The factors* $F_k(\lambda)$ *for* $k = 1,\dots,n-1$ *can be written as:*

$$(17)\qquad \begin{aligned} F_k(\lambda) &= A_{k,0}F_1(\lambda) + A_{k,1}\int_0^{\lambda} F_1(\lambda)F_1(\lambda-\kappa)d\kappa + \\ &+ A_{k,2}\int_0^{\lambda}\Big(\int_0^{\kappa} F_1(t_1)F_1(\kappa-t_1)dt_1\Big)F_1(\lambda-\kappa)d\kappa + \\ &+ A_{k,3}\int_0^{\lambda}\Big(\int_0^{\kappa}\Big(\int_0^{t_1} F_1(t_2)F_1(t_1-t_2)dt_2\Big)F_1(\kappa-t_1)dt_1 F_1(\lambda-\kappa)d\kappa \\ &+ \dots A_{k,k-1}\underbrace{\int_0^{\lambda}\Big(\int_0^{\kappa}\dots}_{(k-1)\ \text{integrals}}\Big)F_1(\lambda-\kappa)d\kappa, \end{aligned}$$

where $F_1(\lambda)$ *has the form (10) and coefficients* $A_{k,i}$ $i = 0,1,\dots$ $\dots,k-1$ *are*

$$(18)\qquad \begin{aligned} A_{2,0} &= 1 + \alpha_{m,1}/Q, \\ A_{2,1} &= 1/Q, \\ A_{k,k-1} &= 1/Q^{k-1}, \\ A_{k,0} &= 1 + \alpha_{m,1}/Q + \alpha_{m,1}^2/Q^2 + \dots + \alpha_{m,1}^{k-1}/Q^{k-1}, \\ A_{k,i} &= (A_{k-1,i-1} + \alpha_{m,1}A_{k-1,i})/Q. \end{aligned}$$

Proof. We start from the solution of equation (4) on the subinterval $[T_{k-1},T_k]$, $k = 2,\dots,n$

$$x_k(\lambda) = x(\lambda) + \frac{e^{-(k-1)hs}}{Q}\int_0^\lambda F_{k-1}(\kappa)x(\lambda-\kappa)d\kappa,$$

and its derivatives:

$$x_k'(\lambda) = x'(\lambda) + \frac{e^{-(k-1)hs}}{Q}\Big(x(0)F_{k-1}(\lambda) + \int_0^\lambda F_{k-1}(\kappa)x'(\lambda-\kappa)d\kappa\Big)$$

$$x_k''(\lambda) = x''(\lambda) + \frac{e^{-(k-1)hs}}{Q}\Big(x'(0)F_{k-1}(\lambda) + \int_0^\lambda F_{k-1}(\lambda)x''(\lambda-\kappa)d\kappa\Big)$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$x_k^{(m)}(\lambda) = x^{(m)}(\lambda) + \frac{e^{-(k-1)hs}}{Q}\Big(x^{(m-1)}(0)F_{k-1}(\lambda) + \int_0^\lambda F_{k-1}(\kappa)x^{(m)}(\lambda-\kappa)d\kappa\Big).$$

In the last relations we used the fact that $x(0) = 0$, $x'(0) = 0, \dots, x^{(m-2)}(0) = 0$. Multiplying each line with the coefficients $\alpha_{\mu,1}$, respectively, for $\mu = 0,\dots,m$, we get:

$$\sum_{\mu=0}^{m}\alpha_{\mu,1}x_k^{(\mu)}(\lambda) = \sum_{\mu=0}^{m}\alpha_{\mu,1}x^{(\mu)}(\lambda) + \frac{\alpha_{m,1}}{Q}H_{k-1}F_{k-1}(\lambda) + \tag{19}$$
$$+ \frac{e^{-(n-1)hs}}{Q}\int_0^\lambda F_{k-1}(\kappa)\Big(\sum_{\mu=0}^{m}\alpha_{\mu,1}x^{(\mu)}(\lambda-\kappa)d\kappa\Big).$$

From (19) follows:

$$F_k(\lambda) = F_1(\lambda) + \frac{\alpha_{m,1}}{Q}F_{k-1}(\lambda) + \frac{1}{Q}\int_0^\lambda F_{k-1}(\kappa)F_1(\lambda-\kappa)d\kappa. \tag{20}$$

After using (20), by mathematical induction, one gets (18).

Lemma 2. *The factors* $\tilde{F}_k(\lambda)$ *can be written as:*

$$\tilde{F}_k(\lambda) = A_{k,0}\tilde{F}_1(\lambda) + A_{k,1}\int_0^{\lambda}\tilde{F}_1(\lambda)\tilde{F}_1(\lambda-\kappa)d\kappa +$$

$$+ A_{k,2}\int_0^{\lambda}\Big(\int_0^{\kappa}\tilde{F}_1(t_1)\tilde{F}_1(\kappa-t_1)dt_1\Big)\tilde{F}_1(\lambda-\kappa)d\kappa +$$

(21)
$$+ A_{k,3}\int_0^{\lambda}\Big(\int_0^{\kappa}\Big(\int_0^{t_1}\tilde{F}_1(t_1)\tilde{F}_1(t_1-t_2)dt_2\Big)\tilde{F}_1(\kappa-t_1)dt_1\Big)\cdot$$

$$\cdot\ \tilde{F}_1(\lambda-\kappa)d\kappa + \dots +$$

$$+ A_{k,k-1}\underbrace{\int_0^{\lambda}\Big(\int_0^{\kappa}(\dots}_{(k-1)\ \text{integrals}}\Big)\tilde{F}_1(\lambda-\kappa)d\kappa,$$

where $\tilde{F}_1(\lambda)$ *has the form (12) and the coefficients* $A_{k,i}$, $i = 0,\dots,k-1$ *are of the form (18).*

The proof is analogous as in the previous Lemma.

The error of approximation.

If ω_j is given in (6), let us introduce the following denotations:

$$\omega_j^{k+\mu} = \{v_{j,k+\mu}(t)\},\quad V_{j,k+\mu}(T) = v_{j,k+\mu}(t)|_{t=T}$$

(22)
$$W_k(T) = \sum_{j=1}^{m} b_j\Big(\sum_{\mu=0}^{m}\alpha_{\mu,1}V_{j,k+\mu}(T)\Big).$$

Using (10) and (22), one can write:

(23)
$$F_1(\lambda) = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}W_k(T)$$

and analogously:

(24) $$\tilde{F}_1(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \tilde{W}_k(T),$$

where:

(25)
$$\tilde{W}_k(T) = \sum_{j=1}^{m} b_j \left(\sum_{=0}^{m} \alpha_{\mu,1} \tilde{V}_{j,k+\mu}(T) \right),$$

$$\tilde{V}_{j,k+\mu}(T) = \tilde{\vartheta}_{j,k+\mu}(t)\Big|_{t=T},$$

$$\{\tilde{v}_{j,k+\mu}(t)\} = \tilde{\omega}_j^{k+\mu}$$

and $\tilde{\omega}_j$ is given in (7).

In order to estimate the difference $|F_k(\lambda) - \tilde{F}_k(\lambda)|$, let us prove:

Lemma 3. *If $F_1(\lambda)$ is given by (23) then:*

(26) $$\int_0^{\lambda} F_1(\kappa)F_1(\lambda-\kappa)d\kappa = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{\lambda^{i+j+1}}{(i+j+1)!} W_i(T)W_j(T) \equiv: B_1(\lambda)$$

where $W_i(T), W_j(T)$ for $i,j = 0,1,\ldots,$ are of the form (23):

Proof. It is known that:

(27) $$\int_0^{\lambda} \kappa^{\alpha-1}(\lambda-\kappa)^{\beta-1}d\kappa = \frac{\Gamma(\alpha)\cdot\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \lambda^{\alpha+\beta-1} .$$

From this relation and relation (23) follows (26).

Lemma 4. *If $F_1(\lambda)$ is given by (23), then*

(28) $$\int_0^{\lambda} \left(\int_0^{t_1} \left(\int_0^{t_2} \ldots \int_0^{t_{k-2}} F_1(t_{k-1})F_1(t_{k-2}-t_{k-1})dt_{k-1}\right)\cdot\right.$$

$$\cdot F_1(t_{k-3} - t_{k-2})dt_{k-2}) \ldots F_1(\lambda-t_1)dt_1 =$$

$$= \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \ldots \sum_{i_k=0}^{\infty} \frac{\lambda^{i_1+i_2+\ldots+i_k+k}}{(i_1+i_2+\ldots+i_k+k)!} W_{i_1}(T)W_{i_2}(T) \ldots$$

$$\ldots W_{i_k}(T) \equiv : B_{k-1}(\lambda)$$

where $W_{i_1}(T), \ldots, W_{i_k}(T)$ *for* $i_1,\ldots,i_k = 0,1,\ldots,$ *are of the form (23).*

The proof follows from relation (26).

Corollary. *If* $\tilde{F}_1(\lambda)$ *is given by (24) then:*

$$(29)\qquad \int_0^\lambda\Big(\int_0^{t_1}\Big(\int_0^{t_2}\Big(\ldots\Big(\int_0^{t_{k-2}}\tilde{F}_1(t_{k-1})\tilde{F}_1(t_{k-2}-t_{k-1})dt_{k-1}\Big)\cdot$$

$$\cdot\,\tilde{F}_1(t_{k-3}-t_{k-2})dt_{k-2}\Big)\ldots\tilde{F}_1(\lambda-t_1)dt_1 =$$

$$= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\ldots\sum_{i_k=0}^{\infty}\frac{\lambda^{i_1+i_2+\ldots+i_k+k}}{(i_1+i_2+\ldots+i_k+k)!}\tilde{W}_{i_1}(T)\ldots$$

$$\ldots\tilde{W}_{i_k} \equiv : \tilde{B}_{k-1}(\lambda)$$

where $\tilde{W}_{i_1}(T),\ldots \tilde{W}_{i_k}(T)$ *for* $i_1,\ldots,i_r = 0,1,\ldots$ *are of the form (25).*

Using (28) and (29), one can write:

$$(30)\qquad F_k(\lambda) = A_{k,0}F_1(\lambda) + A_{k,1}B_1(\lambda) + A_{k,2}B_2(\lambda) + \ldots$$

$$\ldots + A_{k,k-1}B_{k-1}(\lambda)$$

and analogously:

$$(31)\qquad \tilde{F}_k(\lambda) = A_{k,0}\tilde{F}_1(\lambda) + A_{k,1}\tilde{B}_1(\lambda) + \ldots + A_{k,k-1}\tilde{B}_{k-1}(\lambda)$$

The following estimations are going to be needed latter:

(32)
$$|c_{i,j}| \leqslant M_j \; \rho_j^i$$
$$|\ell \sum_{i=0} c_{i,j} \; \ell^{i/q}| \leqslant_T P_j(T)\ell \; .$$

Lemma 5. *If* $W_k(T)$ *is given by (22) and* $\tilde{W}_k(T)$ *is given by (25) then:*

(33)
$$|W_k(T) - \tilde{W}_k(T)| \leqslant \sum_{\mu=0}^{m} |\alpha_{\mu,1}| \sum_{j=1}^{m} |b_j|\rho_j^{i_0+1}P_j^{k+\mu}(T)(k+\mu) \cdot$$
$$\cdot \frac{T^{\frac{i_0+1}{q_i} - \frac{p_j}{q_j}(k+\mu) + 1}}{\Gamma(\frac{i_0+1}{q_j} - \frac{p_j}{q_j}(k+\mu) + 2)} \equiv R_k(T)$$

where $P_j(T)$, *and* ρ_j *are given by (32).*

Proof.
$$|\sum_{j=1}^{m} b_j \sum_{\mu=0}^{m} \alpha_{\mu,1} \; \omega_j^{k+\mu} - \sum_{j=1}^{m} b_j \sum_{\mu=0}^{m} \alpha_{\mu,1}\tilde{\omega}_j^{k+\mu}| \leqslant_T$$
$$\leqslant_T \sum_{\mu=0}^{m} |\alpha_{\mu,1}| \sum_{j=1}^{m} |b_j|\rho_j^{i_0+1} \; P_j^{k+\mu}(T)(k+\mu)\cdot$$
$$\cdot \; \ell^{\frac{i_0+1}{q_j} - \frac{p_j}{q_j}(k+\mu) + 2}.$$

Since the last estimate holds for any $t \leqslant T$, the relation (33) is satisfied.

Now, using Lemma 5, we get:

(34)
$$|W_{i_1}(T)W_{i_2}(T) - \tilde{W}_{i_1}(T)\tilde{W}_{i_2}(T)| \leqslant R_{i_1}(T)\tilde{\tilde{W}}_2(T) +$$
$$+ \tilde{\tilde{W}}_{i_1}(T)R_{i_2}(T) \equiv : Q_{i_2}(T), \qquad i_1,i_2 \in N$$

where $\tilde{\tilde{W}}_k$ is defined by:

$$\left|\sum_{\mu=0}^{m} \alpha_{\mu,1} \sum_{j=1}^{m} b_j \omega^{k+\mu}\right| \leqslant_T \tilde{\tilde{W}}_k(T)\ell$$

and $R_{i_1}(T)$ and $R_{i_2}(T)$ are given by (33).

Similary, we obtain Q_{i_k} as:

(35) $$|W_1(T) \ldots W_{i_k}(T) - \tilde{W}_1(T) \ldots \tilde{W}_{i_k}(T)| \leqslant Q_{i_k}(T).$$

Now, we can find the difference between $F_k(\lambda)$ and $\tilde{F}_k(\lambda)$:

Proposition 1. *If $F_k(\lambda)$ is given by (17) and $\tilde{F}_k(\lambda)$ is given by (21), then:*

(36) $$\begin{aligned}|(F_k(\lambda) - \tilde{F}_k(\lambda))| \leqslant_T\ & |A_{k,0}\ell| \sum_{i=0}^{\infty} \frac{|\lambda|^i}{i!} R_{i_1}(T) + \\ & + |A_{k,1}\ell| \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \frac{|\lambda|^{i_1+i_2+1}}{(i_1+i_2+1)!} Q_{i_2}(T) + \ldots + \\ & + |A_{k,k-1}\ell| \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \ldots \sum_{i_k=0}^{\infty} \frac{|\lambda|^{i_1+i_2+\ldots+i_k+k}}{(i_1+i_2+\ldots+i_k+k)!} Q_{i_k}(T) \\ \equiv\ & : \mathcal{R}_{\varepsilon,k}(\lambda,T)\end{aligned}$$

where $R_{i_1}(T)$, and $Q_{i_k}(T)$ are given bu (33) and (35).

Proof. Using (17), (18) and (21), we have:

$$\begin{aligned}|\ell(F_k(\lambda) - \tilde{F}_k(\lambda))| \leqslant\ & |A_{k,0}\ell||F_1(\lambda) - \tilde{F}_1(\lambda)| + \\ & + |A_{k,1}\ell|\left|\int_0^\lambda F_1(\kappa)F_1(\lambda-\kappa)d\kappa - \int_0^\lambda \tilde{F}_1(\kappa)\tilde{F}_1(\lambda-\kappa)d\kappa\right| + \\ & + |A_{k,2}\ell|\left|\int_0^\lambda \left(\int_0^\kappa F_1(t_1)F_1(\kappa-t_1)dt_1\right)F_1(\lambda-\kappa)d\kappa - \right.\end{aligned}$$

$$- \int_0^\lambda \left(\int_0^\kappa \tilde{F}_1(t_1)\tilde{F}_1(\kappa-t_1)dt_1 \tilde{F}_1(\lambda-\kappa)d\kappa \right| + \ldots +$$

$$+ |A_{k,k-1}\ell|| \int_0^\lambda \left(\int_0^\kappa (\ldots\ )F_1(\lambda-\kappa)d\kappa \right. -$$

$$- \int_0^\lambda \left(\int_0^\kappa (\ldots\ )\tilde{F}_1(\lambda-\kappa)d\kappa \right|$$

and from (28) and (29) follows:

$$|\ell(F_k(\lambda) - F_k(\lambda))| \leqslant_T |A_{k,0}\ell| \left(\sum_{i_1=0}^{\infty} \frac{|\lambda|^{i_1}}{(i_1)!} |W_{i_1}(T) - \right.$$

$$\left. - \tilde{W}_{i_1}(T)| \right) + |A_{k,1}\ell| \sum_{i=0}^{\infty} \sum_{i=0}^{\infty} \frac{\lambda^{i_1+i_2+1}}{(i_1+i_2+1)!} W_{i_1}(T)W_{i_2}(T) -$$

$$- \tilde{W}_{i_1}(T)\tilde{W}_{i_2}(T)| + \ldots +$$

$$+ |A_{k,k-1}\ell| \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \ldots \sum_{i_k=0}^{\infty} \frac{\lambda^{i_1+i_2+\ldots+i_k+k}}{(i_1+i_2+\ldots+i_k)!} \cdot$$

$$\cdot |W_{i_1}(T)W_{i_2}(T) \ldots W_{i_k}(T) - \tilde{W}_{i_1}(T) \ldots \tilde{W}_{i_k}(T)|.$$

Finally, using (34) and (35) we have:

$$|\ell(K_k(\lambda) - \tilde{F}_k(\lambda))| \leqslant_T |A_{k,o}\ell| \sum_{i_1=0}^{\infty} \frac{|\lambda|^{i}}{i_1!} R_{i_1}(T) +$$

$$+ |A_{k,1}| \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \frac{|\lambda|^{i_1+i_2+1}}{(i_1+i_2+1)!} Q_{i_2}(T) + \ldots +$$

$$+ |A_{k,k-1}\ell| \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \ldots \sum_{i_k=0}^{\infty} \frac{|\lambda|^{i_1+i_2+\ldots+i_k+k}}{(i_1+i_2+\ldots+i_k+k)!} Q_{i_k}(T).$$

Proposition 2. *If* $x_n(\lambda)$ *is given by (13) and* $\tilde{x}_n(\lambda)$ *is given by (15) then the error of approximation is of the form:*

$$|x_n(\lambda) - \tilde{x}_n(\lambda)| \leqslant_T x_\varepsilon(\lambda)\ell + \lambda(\tilde{\tilde{R}}_{\varepsilon,n-1}(\lambda,T)\tilde{\tilde{x}}(\lambda)\ell + \\ + \tilde{\tilde{F}}_n(\lambda)\tilde{\tilde{x}}_\varepsilon(\lambda)\ell)$$

where

$$|x(\lambda) - \tilde{x}(\lambda)| \leqslant_T x_\varepsilon(\lambda)\ell$$

$$\tilde{\tilde{R}}_{\varepsilon,n-1}(\kappa,T) = \max_{0\leqslant\kappa\leqslant\lambda} R_{\varepsilon,n-1}(\kappa,T)$$

$$\tilde{\tilde{F}}(\lambda) = \max_{0\leqslant\kappa\leqslant\lambda} \tilde{F}(\kappa)$$

$$\max_{0\leqslant\lambda\leqslant\kappa} |x(\lambda-\kappa)| \leqslant \tilde{\tilde{x}}(\lambda)$$

$$\tilde{\tilde{x}}_\varepsilon(\lambda) = \max_{0\leqslant\kappa\leqslant\lambda} x_\varepsilon(\lambda-\kappa)$$

and $R_{\varepsilon,n-1}(\kappa,T)$ *is given by (36).*

Proof. Using (13) and (15) we have:

$$|x_n(\lambda) - \tilde{x}_n(\lambda)| \leqslant |x(\lambda) - \tilde{x}(\lambda)| + \\ + \int_0^\lambda |e^{-s(n-1)h}(F_{n-1}(\kappa) - \tilde{F}_{n-1}(\kappa))x(\lambda-\kappa)|d\kappa \\ + \int_0^\lambda |e^{-s(n-1)h}\tilde{F}_{n-1}(\kappa)(x(\lambda-\kappa) - \tilde{x}(\lambda-\kappa))|d\kappa \leqslant_T$$

$$\leqslant_T x_\varepsilon(\lambda)\ell + \lambda(\tilde{\tilde{R}}_{\varepsilon,n-1}(\lambda,T)\tilde{\tilde{x}}(\lambda)\ell + \tilde{\tilde{F}}_n(\lambda)\tilde{\tilde{x}}_\varepsilon(\lambda)\ell).$$

Numerical example. Let us observe the following partial differential equation:

(38) $$\frac{\partial^2 x(\lambda,t)}{\partial\lambda\partial t} - \frac{\partial x(\lambda,t)}{\partial\lambda} - x(\lambda,t) = 0$$

with conditions:

(39) $$\frac{\partial x(\lambda,0)}{\partial\lambda} = 0,\ \lambda > 0$$

$$x(0,t) = 1,\quad t > 0.$$

In the field M, equation

(40) $$(s-1)x'(\lambda) - x(\lambda) = 0$$

corresponds to the equation (38) with conditions (39). The solution of equation (40) is:

(41) $$x(\lambda) = \ell exp(\lambda\omega),\ \omega = \sum_{i=0}^{\infty} \ell^{i+1},$$

while the approximate solution is of the form:

(42) $$x(\lambda) = \ell exp(\lambda\tilde{\omega}),\ \tilde{\omega} = \sum_{i=0}^{i_0} \ell^{i+1}.$$

After dividing interval $[0,T]$ on n subintervals the solution on the interval $[T_{n-1},T]$ can be written as:

$$x_n(\lambda) = x(\lambda) + \frac{e^{-(n-1)hs}}{Q}\int_0^\lambda F_{n-1}(\kappa)x(\lambda-\kappa)d\kappa$$

and the approximate one is of the form:

$$\tilde{x}_n(\lambda) = \tilde{x}(\lambda) + \frac{e^{-(n-1)hs}}{Q}\int_0^\lambda \tilde{F}_{n-1}(\kappa)\tilde{x}(\lambda-\kappa)d\kappa,$$

where $Q = \ell(s-1)$, $x(\lambda)$ is of the form (41), $\tilde{x}(\lambda)$ is of the form (42), $F_{n-1}(\lambda)$ and $\tilde{F}_{n-1}(\kappa)$ are given by (17) and (21), respectively.

The following table shows the dependence of the error of approximation on the number of subintervals. For $i_0 = 11$, $\lambda = 1$ we have:

T \ n	1	2	5
0,5	$4,07970 \cdot 10^{-5}$	$3,78263 \cdot 10^{-6}$	$5,798 \cdot 10^{-9}$
1,0	$5,74297 \cdot 10^{-2}$	$5,417665 \cdot 10^{-4}$	$1,73472 \cdot 10^{-6}$
2,0	$5,09084 \cdot 10^{1}$	$7,15463 \cdot 10^{0}$	$4,16679 \cdot 10^{-3}$

REFERENCES

[1] Mikusiński, J., *Operator Calculus*, *Warszawa*, *1959*.

[2] Stanković, B., *Approximate solution of the operator linear differential equation I*, *Publ. Inst. Math.*, *21(35)*, *1977*, *185 - 196*.

[3] Takači, Dj., *The Approximate Solution of the differential equation in many steps*, *Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak.*, *Ser. Mat. 13*, *(1983)*,

REZIME

O OBLIKU PRIBLIŽNOG REŠENJA JEDNE PARCIJALNE DIFERENCIJALNE JEDNAČINE

U ovom radu se posmatra linearna parcijalna diferencijalna jednačina sa konstantnim koeficijentima u polju operatora Mikusińskog. Konstruisan je novi oblik približnog rešenja i ocenjena je greška.

Received by the editors June 16, 1986.