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ON THE NUMBER OF MONOTONE FUNCTIONS OF $P^2_{k,3}$

Ratko Tošić

Prirodno-matematički fakultet, Institut za matematiku, 21000 Novi Sad, Dr Ilije Djuričića 4, Jugoslavija

ABSTRACT

In this paper a formula is given for the number of monotone functions of $P^2_{k,3} = \{f \mid E^2_k \to E_3\}$, where $E_m = \{0,1,\ldots, m-1\}$, for any natural m.

1. DEFINITIONS AND NOTATION

Let X denote a finite and nonempty set of symbols, i.e. an alphabet. By X^n we shall denote the set of all strings of the length n over the alphabet X, i.e. $X^n = \{x_1x_2\ldots x_n \mid x_1,x_2,\ldots,x_n \in X\}$, the only element of X^0 being the empty string (the string of length 0). The set of all the finite strings over the alphabet X is $X^* = \bigcup_{i\geq 0} X^i$.

We shall also use some special notations:

$A = \{0,1\}$;

$B = \{0,1,2,3,\}$;

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$l_j(a)$ - the number of j's in the string $a \in A^*$, for $j \in A$;

$l_j(b)$ - the number of j's in the string $b \in B^*$, for $j \in B$.

If S is a set, then $|S|$ is the cardinality of S.

We shall denote by $P^2_{k,3}$ the set of functions mapping the set $E^2_k = \{0,1,\ldots,k-1\}^2$ into the set $E_3 = \{0,1,2\}$, i.e. $P^2_{k,3} = \{f\colon E^2_k \to E_3\}$.

A function $f \in P^2_{k,3}$ is said to be monotone if $x \le y$ implies $f(x) \le f(y)$; where $x = (x_1,x_2,\ldots,x_n) \le$ $\le (y_1,y_2,\ldots,y_n) = y$ iff $x_1 \le y_1$, $x_2 \le y_2,\ldots,x_n \le y_n$, and $0 < 1 < 2 < \ldots < m-1$ in E_m, for any natural m.

Let M denote the set of all monotone functions of $P^n_{k,3}$.

2. RESULT

The number of all symmetric monotone functions of n variables over the three-valued logic algebra, i.e. the number of functions of the set $P^n_{3,3} = \{f\colon E^4_3 \to E_3\}$ is determined in [2]. In [1] it is also proved that this number is $\binom{2n+2}{n+1}$, by establishing the bijection between the set of all symmetric monotone functions of $P^n_{3,3}$ and the set of all symmetric monotone functions of $P^2_{n+1,3}$.

Now, our aim is to determine the number of all monotone functions of $P^2_{k,3}$. In Figure 1, the set E^2_k is represented as the lattice of all the points (p,q), $0 \le p, q \le k-1$.

We shall also consider the lattice of all points $(p-\frac{1}{2}, q-\frac{1}{2})$, $0 \le p,q \le k$.

A decreasing path from $(-\frac{1}{2}, k+\frac{1}{2})$ to $(k+\frac{1}{2}, -\frac{1}{2})$ is a set of edges of this lattice, which at each point either increases in p or decreases in q. Label each edge of such a path by 0 if it increases in p and by 1 if it decreases in q. So, there is a bijection between the set of all decreasing paths beginning at $(-\frac{1}{2}, k+\frac{1}{2})$ and ending at $(k+\frac{1}{2},$

$-\frac{1}{2})$ and the set of all strings of A^{2k} consisting of k 1s and k 0s. In Figure 1, two such paths s_1 and s_2, for $k = 7$, are drawn and corresponding strings are 10110010110010 and 00101100101011, respectively.

THEOREM.

$$|M \cap P^2_{k,3}| = \frac{1}{k+1}\binom{2k}{k}\binom{2k+1}{k}.$$

PROOF. Any function $f\colon E^2_k \to E_3$ is completely determined by three sets

$$T_i = \{(p,q) \mid (p,q) \in E^2_k,\ f(p,q) = i\}, \quad \text{for } i = 0,1,2.$$

However, the sets T_o, T_1 and T_2, corresponding to a monotone function $f\colon E^2_k \to E_3$ are separated by two decreasing paths s_1 and s_2 beginning at $(-\frac{1}{2}, k+\frac{1}{2})$ and ending at $(k+\frac{1}{2}, -\frac{1}{2})$, and such that none of the points of s_2 are below s_1. On the other hand, such two paths always determine a monotone function $f\colon E^2_k \to E_3$, by specifying corresponding sets T_o, T_1 and T_2.

So, there is a bijection between the set of all monotone functions $f\colon E^2_k \to E_3$ and the set of all pairs of strings $a_1a_2\ldots a_{2k}$, $a'_1a'_2\ldots a'_{2k} \in A^{2k}$, such that

$$l_o(a_1a_2\ldots a_{2k}) = l_1(a_1a_2\ldots a_{2k}) = l_o(a'_1a'_2\ldots a'_{2k}) =$$
$$= l_1(a'_1a'_2\ldots a'_{2k}) = k,$$

and

$$l_1(a'_1a'_2\ldots a'_r) \le l_1(a_1a_2\ldots a_r),$$

for each $r = 1,2,\ldots,2k$.

However, the number of such pairs of strings is equal to the number of strings $b_1b_2\ldots b_{2k} \in B^{2k}$ such that

(1) $$l_1(b_1b_2\ldots b_{2k}) = l_2(b_1b_2\ldots b_{2k}),$$

(2) $$l_o(b_1b_2\ldots b_{2k}) = l_3(b_1b_2\ldots b_{2k}),$$

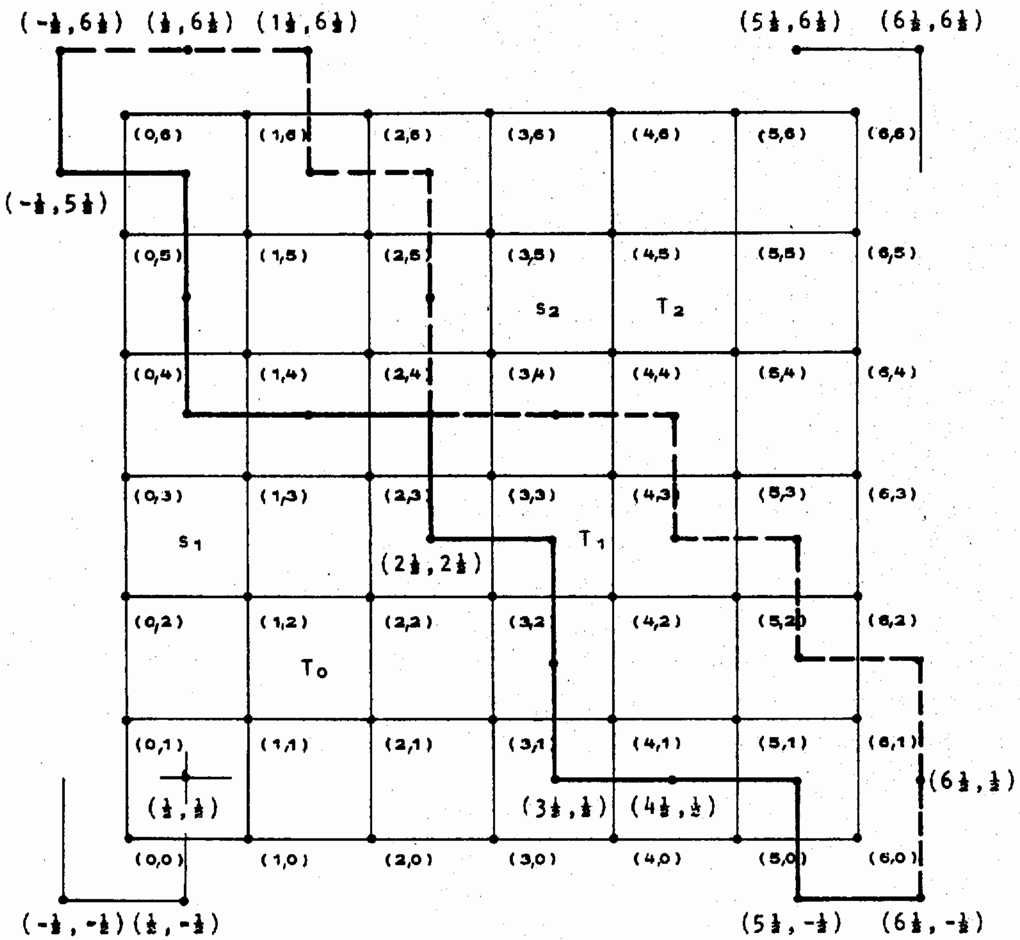
(-½,6½) (½,6½) (1½,6½)
(5½,6½) (6½,6½)
(-½,5½)
(0,6) (1,6) (2,6) (3,6) (4,6) (5,6) (6,6)
(0,5) (1,5) (2,5) (3,5) (4,5) (5,5) (6,5)
S2 T2
(0,4) (1,4) (2,4) (3,4) (4,4) (5,4) (6,4)
(0,3) (1,3) (2,3) (3,3) (4,3) (5,3) (6,3)
S1 T1
(2½,2½)
(0,2) (1,2) (2,2) (3,2) (4,2) (5,2) (6,2)
T0
(0,1) (1,1) (2,1) (3,1) (4,1) (5,1) (6,1)
(½,½) (3½,½) (4½,½) (6½,½)
(0,0) (1,0) (2,0) (3,0) (4,0) (5,0) (6,0)
(-½,-½) (½,-½) (5½,-½) (6½,-½)

Figure 1

and

(3) $l_1(b_1b_2\ldots b_r) \leq l_2(b_1b_2\ldots b_r)$, for each $r=1,2,\ldots\ldots 2k$.

The corresponding bijection can be established by taking $b_i = 2a_i' + a_i$ $(i = 1,2,\ldots,2k)$.

Denote by $B(k)$ the set of strings belonging to B^{2k} and satisfying (1), (2) and (3). Let

$$B_i(k) = \{b|b \in B(k),\ l_0(b)=l_3(b) = i\}, i = 0,1,\ldots,k.$$

Then,

$$|B_i(k)| = \frac{1}{k+1}\binom{2k}{k}\binom{k}{i}\binom{k+1}{i};$$

hence,

$$|M \cap P^2_{k,3}| = |B(k)| = \sum_{i=0}^{k} |B_i(k)| =$$

$$= \frac{1}{k+1}\binom{2k}{k}\sum_{i=0}^{k}\binom{k}{i}\binom{k+1}{ii} = \frac{1}{k+1}\binom{2k}{k}\binom{2k+1}{k}.\square$$

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REZIME

O BROJU MONOTONIH FUNKCIJA U $P^2_{k,3}$

U radu je data formula za broj monotonih funkcija u skupu $P^2_{k,3} = \{f|E^2_k \in E_3\}$, gde je $E_m = \{0,1,\ldots,m-1\}$, za proizvoljan prirodan broj m.

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