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SEARCH NUMBER OF THE CARTESIAN PRODUCT OF GRAPHS

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ABSTRACT

An upper bound for the search number of the Cartesian product $G_1 + G_2$ is determined, where the search numbers of the graphs G_1 and G_2 are $s(G_1)$ and $s(G_2)$, respectively. Using this, some estimates for the search numbers of n-cubes are obtained, for n natural.

Let G be a finite connected graph without loops or multiple edges. We may assume that G is embedded in R^3 so that its vertices $v_1, v_2, \ldots, v_p$ are represented by distinct points, and its edges are represented by closed line segments in R^3 which intersect only at the vertices of G. Regarded as a subset of R^3, G is a topological space with the relative topology. Then G is a compact locally connected metric space in which every connected set is arcwise connected.

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The following definitions are from Parsons ([1], [2]).

Definition 1. *A search plan for* G *is a family* $\mathcal{F} = \{f_i \mid 1 \le i \le k\}$ *of continuous functions* $f_i : [0,\infty) \to G$, *such that for every continuous function* $g : [0,\infty) \to G$ *there exists* $t_0 \in [0,\infty)$ *and an* $i \in \{1,2,\dots,k\}$ *such that* $g(t_0) = f_i(t_0)$.

We think of $g(t)$ and $f_i(t)$ as the positions at time t of a lost man and the ith searcher in the cave represented by graph G, in which the searchers and the lost man move contինaously.The searchers must proceed according to a predetermined plan which will capture the lost man even if he were an arbitrarily fast, invisible evader who, clairvoyant, knows the searchers' every move. Then a search plan must catch any possible evader in a finite time.

Definition 2. *The search number* s(G) *of* G *is the minimum cardinality of all search plans for* G.

Our problem is to give an upper bound for the search number of the Cartesian product of given graphs for which the search numbers are known.

Let $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ be two finite connected graphs. We denote by $G_1 + G_2$ the Cartesian product G of the graphs G_1 and G_2 (somewhere called the sum), i.e.

$$G = G_1 + G_2 = (V,E),$$

where

$$V = V_1 \times V_2$$

and

$$E = \{(x_1,y_1)(x_2,y_2) \mid (x_1x_2 \in E_1 \text{ and } y_1 = y_2)$$
$$\text{or } (x_1 = x_2 \text{ and } y_1y_2 \in E_2)\}.$$

Theorem 1. *If* $G_1 = (V_1,E_1)$ *and* $G_2 = (V_2,E_2)$ *are two finite connected graphs, then*

$$s(G_1 + G_2) \leq \min\{|V_1|s(G_2),\ |V_2|s(G_1)\} + 1.$$

Proof. Let $V_1 = \{x_1,x_2,\dots,x_p\}$, $V_2 = \{y_1,y_2,\dots,y_k\}$, $s(G_1) = m$, $s(G_2) = n$. Suppose that $pn \leq qm$ and that we have at our disposal $pn + 1$ searchers. Denote these $pn + 1$ searchers by

$$C_0;\ C_1^1,C_1^2,\dots,C_1^p;\ C_2^1,C_2^2,\dots,C_2^p;\dots;\ C_n^1,C_n^2,\dots,C_n^p.$$

Let the vertices $y_{k_1},y_{k_2},\dots,y_{k_n}$ (not necessarily different) make a starting position for a search plan in G_2. (Of course, if $s(G) = n$, for a graph G, then any set of at most n points of G, considered as a subset of R^3, can serve a starting position for a search plan, but the theorem also holds for digraphs). First, we place all searchers C_j^i ($i \in \{1,2,\dots$ $\dots,p\}$, $j \in \{1,2,\dots,n\}$) on the starting vertices in such a way that the searcher C_j^i occupies the vertex (x_i,y_{k_j}). Now, all the vertices of the copies $G_{y_{k_1}},G_{y_{k_2}},\dots,G_{y_{k_n}}$ of the graph G_1 are occupied. (G_{y_k} is the copy of G_1 induced by the vertices (x_1,y_k), (x_2,y_k), ..., (x_p,y_k) of the graph $G = G_1 + G_2$. Similarly, G_{x_h} is the copy of G_2 induced by the vertices (x_h,y_1), $(x_h,y_2),\dots,(x_h,y_q)$ of the graph $G = G_1 + G_2$). Then the searcher C_0 traverses all the edges of these copies. If the lost man is on some of these edges, he will be found. If not, we begin by simultaneous realizations of p search plans in the copies $G_{x_1},G_{x_2},\dots,G_{x_p}$ of G_2; the searchers $C_1^i,C_2^i,\dots,C_n^i$ search in the copy G_{x_i}. Each time the searchers C_j^i ($i \in \{1,2,\dots,p\}$, $j \in \{1,2,$ $\dots n\}$) occupy all the vertices of some copies $G_{y_{t_1}},G_{y_{t_2}},\dots,G_{y_{t_n}}$ (not necessarily different) of G_1, they make a pause during which the searcher C_0 traverses all the edges of these copies. The search continues in this way. So, the lost man will be found either in a copy G_{x_i} of G_2 by the searchers $C_1^i,C_2^i,\dots,C_n^i$ or in a copy G_{y_r} of G_2 by the searcher C_0. □

Theorem 1 can be generalized in the following way

Theorem 2. *Let* $G_1 = (V_1,E_1)$, $G_2 = (V_2,E_2),\dots,$ $G_m =$

$= (V_m, E_m)$ *be finite connected graphs.*
Then

$$s(G_1 + G_2 + \dots + G_m) \le \min_{1\le i\le m} \{s(G_i) \prod_{\substack{1\le j\le m \\ j\ne i}} |V_j|\} + 1.$$

Proof. Similar as for Theorem 1. □

From Theorem 2, we obtain an upper bound for the search number of the n-dimensional cube Q_n.

Corollary 1.

$$s(Q_n) \le 2^{n-1} + 1.$$

For $n = 1$ *and* $n = 2$, *the strict inequality holds. Namely,* $s(Q_1) = 1$ *and* $s(Q_2) = 2$.

For $n = 3$, *the equality holds, i.e.* $s(Q_3) = 5$, *but a rigorous proof requires some care.*

Similarly, for the search number of the Cartesian product of paths we obtain:

Corollary 2.

$$s(P_{n_1} + P_{n_2} + \dots + P_{n_m}) \le \frac{n_1 n_2 \dots n_m}{\max\{n_1, n_2, \dots, n_m\}} + 1.$$

REFERENCES

[1] T.D. Parsons: *Pursuit-evasion in a graph*, Lecture Notes in Math., 642, Springer, Berlin, (1978) 426 - - 441.

[2] T.D. Parsons: *The search number of a connected graph* Proc. 9th South-Eastern Conf. on Combinatorics, Graph Theory and Computing, (1978), 549 - 554.

REZIME

ISTRAŽNI BROJ KARTEZIJEVOG PROIZVODA GRAFOVA

U radu je odredjena jedna gornja granica za s-broj Dekartovog proizvoda $G_1 + G_2$, za poznate s-brojeve $s(G_1)$ i $s(G_2)$. Kao posledica dobijena je procena za broj $s(Q_n)$, n je prirodan broj, gde je Q_n n-dimenzionalna kocka.

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