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ON UNAVOIDABLE SUBDIGRAPHS OF TOURNAMENTS

Vojislav Petrović

University of Novi Sad, Faculty of Science,
Institute of Mathematics, Dr I. Djuričića 4,
21000 Novi Sad, Yugoslavia

ABSTRACT

A digraph D is n-unavoidable if each tournament on n vertices contains a subdiagraph isomorphic to D. It is proved a that the diagraph H(n,i) defined as a simple n-path $v_1 \to v_2 \to \dots \to v_n$ with an aditional are v_1v_i $(3 \le i \le n)$, is n-unavoidable for each $n(n \ge 4)$ and $i = 4$. So are H(n,3) and H(n,n-1) for $n \ge 4$, excluding two particular cases.

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The terminology used in the paper is that of [3], except as noted. A digraph D is said to be n-*unavoidable* if each tournament on n vertices contains a subdigraph isomorphic to D. Let H(n,i) be a simple n-path $v_1 \to v_2 \to \dots \to v_n$ with an additional arc v_1v_i $(3 \le i \le n)$. The following two results are well known.

(A) (Rédei, [4]) the Hamiltonian path is n-unavoidable for each $n(n \ge 2)$.

(B) (Grünbaum, [1, p. 211]) H(n,n) - Hamiltonian bypass is n-unavoidable for each n $(n \ge 3)$, except for two tournaments T_3^* and T_5^* (Fig. 1).

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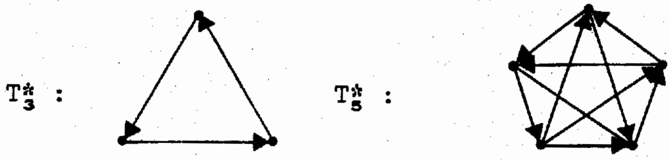

Fig. 1.

At the Sixth Yugoslav Seminar on Graph Theory, Zagreb 1986, V. Sós proposed the

Conjecture. $H(n,i)$ *is* n-*unavoidable for each* n ($n \ge 5$) *and each* i ($4 \le i \le n-1$).

We shall prove the conjecture for $i = 4$, and show that $H(n,3)$ and $H(n,n-1)$ are also n-unavoidable for $n \ge 4$, except for two particular cases.

Theorem 1. $H(n,4)$ *is* n-*unavoidable for each* n ($n \ge 4$).

Proof. By induction on n. For $n = 4$, the theorem follows by (B). If $n = 5$, let T_5 be an arbitrary 5-tournament with vertex set $\{v_1,v_2,v_3,v_4,v_5\}$. First assume that T_5 is not strong. Then there is a unique decomposition of T_5 into strong components $T_5^{(1)} \to T_5^{(2)} \to \ldots \to T_5^{(i)}$, where $i \le 5$, and each vertex of $T_5^{(i)}$ dominantes each vertex of $T_5^{(j)}$ iff $i < j$. If $|V(T_5^{(1)})| < 4$, the assertion follows by (A). If $|V(T_5^{1})| = 4$ then, by (B), $T_5^{(1)}$ has a Hamiltonian bypass, which composed with $T_5^{(2)}$ (obviously $k = 2$ and $T_5^{(2)}$ is a single vertex) gives $H(5,4)$. Now assume that T_5 is strong. Let $v_1 \to v_2 \to \ldots \to v_n$ be its Hamiltonian cycle. Then $v_{i+3} \to v_i$ for each $i \in \{1,2,3,4,5\}$ (all sums are modulo 5). Otherwise, $v_i \to v_{i+1} \to v_{i+2} \to v_{i+3} \to$ $\to v_{i+4}$ is $H(5,4)$ in T_5. But, now, $v_1 \to v_3 \to v_5 \to v_2 \to v_4$ is $H(5,4)$. Therefore, the theorem holds for $n = 5$.

Suppose that $H(n,4)$ is n-unavoidable for some n

($n > 5$), and prove that is $H(n+1,4)$ also. Assume that there is a tournament T_{n+1} on $n+1$ vertices without $H(n+1,4)$ and show that it leads to a contradiction.

Let $\{v_1, v_2, \ldots, v_n, v\}$ be the vertex set of T_{n+1}. By the induction hypothesis, $T_n - v$ has $H(n,4)$. We can assume w.l.g. that it is $v_1 \to v_2 \to \ldots \to v_n$. This clearly forces

(1) $$v \to v_n.$$

Also

(2) $$v \to \{v_4, v_5, \ldots, v_{n-1}\}.$$

Indeed, if $v_i \to v$ ($4 \le i \le n-1$), denote by i_0 the smallest i such that $v_{i_0} \to v$ and $v \to v_{i_0+1}$. Inserting v into the path $v_4 \to v_5 \to v_6 \to \ldots \to v_n$, we get $H(n+1,4)$

$$v_1 \to v_2 \to v_3 \to v_4 \to \ldots \to v_{i_0} \to v \to v_{i_0+1} \to \ldots \to v_n.$$

Next we shall consider two cases.

Case 1. $v \to v_1$. Then

(3) $$v_3 \to v,$$

because of $H(n+1,4)$ - $v \to v_1 \to v_2 \to \ldots \to v_n$. Similarly,

(4) $$v_1 \to v_n,$$

because of $v \to v_4 \to v_5 \to \ldots \to v_n \to v_1 \to v_2 \to v_3$ (by (2)). Further,

(5) $$v_2 \to v_n,$$

because of $v \to v_1 \to v_4 \to v_5 \to \ldots \to v_n \to v_2 \to v_3$. Reasoning in the same way as for (2), we conclude that

(6) $$v_2 \to \{v_4, v_5, \dots, v_{n-1}\}.$$

Subcase 1.1. $v \to v_2$. Then $H(n+1,4) - v_1 \to v_3 \to v \to \to v_2 \to v_4 \to v_5 \to \dots \to v_n$ (by (6)) implies

(7) $$v_3 \to v_1.$$

But this produces $H(n+1,4) - v \to v_2 \to v_3 \to v_1 \to v_4 \to v_5 \to \dots \to \to v_n$, a contradiction.

Subcase 1.2. $v_2 \to v$. $H(n+1,4) - v_2 \to v \to v_1 \to v_3 \to \to v_4 \to \dots \to v_n$ forces (7) and it gives, by (3), $v_3 \to v_1 \to v_2 \to \to v \to v_4 \to v_5 \to \dots \to v_n$, contradicting the assumption.

Case 2. $v_1 \to v$. Then (2) and $v_1 \to v_2 \to v_3 \to v \to v_4 \to \to \dots \to v_n$ imply

(8) $$v \to v_3.$$

Subcase 2.1. $v \to v_2$. Then

(9) $$v_3 \to v_1,$$

because of $v_1 \to v \to v_2 \to v_3 \to \dots \to v_n$. Using (9), we get from $H(n+1,4)$ $v_3 \to v_1 \to v \to v_4 \to \dots \to v_n \to v_2$

(10) $$v_2 \to v_n.$$

Further (10) forces

(11) $$v_2 \to \{v_4, v_5, \dots, v_{n-1}\}.$$

Otherwise, v_2 can be inserted in the path $v_4 \to v_5 \to \dots \to v_n$, producing $H(n+1,4) - v_3 \to v_1 \to v \to v_4 \to \dots \to v_2 \to \dots \to v_n$.

But by (7) and (11), we have $H(n+1,4) - v \to v_3 \to v_1 \to v_2 \to v_4 \to$ $\to \dots \to v_n$.

Subcase 2.2. $v_2 \to v$. Again (9) holds by $v_1 \to v_2 \to$ $\to v \to v_3 \to v_4 \to \dots \to v_n$. But, now, it gives $H(n+1,4) - v_2 \to$ $\to v_3 \to v_1 \to v \to v_4 \to \dots \to v_n$, completing the proof.

The following lemma is important for discussing $H(n,3)$.

Lemma. *Let* $H_n : v_n \to v_{n-1} \to \dots \to v_2 \to v_1$ *be a Hamiltonian path of a tournament* T_n $(n \geq 4)$ *in which arcs* $v_i v_{i+2}$ $(i = 1,2,\dots,n-2)$ *are present. Then there exists a Hamiltonian path of* T_n *starting at* v_1.

Proof. By induction on n. First we check for small n´s.

(a) $n = 4$. Then we have $H_4 : v_1 \to v_3 \to v_2 \to v_4$.

(b) $n = 5$. If $v_2 \to v_5$, then $H_5 : v_1 \to v_3 \to v_2 \to v_5 \to v_4$ and if $v_5 \to v_2$, then $H_5 : v_1 \to v_3 \to v_5 \to v_2 \to v_4$.

(c) $n = 6$. $H_6 : v_1 \to v_3 \to v_2 \to v_4 \to v_6 \to v_5$ is present in T_6.

Now suppose that the lemma is true for all positive integers not greater than n, and prove that it holds for n+1 too.

Case 1. $n = 3k$ $(k \geq 2)$. Then $n+1 = (3(k-1) + 1) + 3$. By the induction hypothesis and (a), there exists a path P_1 : $v_1 \to \dots \to v_{3(k-1)+1}$ containing vertices $v_1, v_2, \dots, v_{3(k-1)+1}$ and the path $P_2 : v_{3(k-1)+1} \to v_{3(k-1)+3} \to v_{3(k-1)+2} \to v_{3(k-1)+4}$. Connecting P_1 and P_2, we get H_{n+1} starting at v_1.

Case 2. $n = 3k+1$ $(k \geq 2)$. Now, $n+1 = (3(k-1)+1) + 4$ and P_1 connected with P_3 : $v_{3(k-1)+1} \to v_{3(k-1)+3} \to \dots \to v_{3(k-1)+4}$ (using (b)) produces again H_{n+1} starting at v_1.

Case 3. $n = 3k+2$ $(k \geq 2)$. Using (c) and connecting P_1 with P_3 : $v_{3(k-1)+1} \to v_{3(k-1)+2} \to \dots \to v_{3(k-1)+4}$, we get H_{n+1} with the starting vertex v_1.

Theorem 2. *$H(n,3)$ is n-unavoidable for each n $(n \geq 4)$, unless T_n is of the type $T_3 \to T^*_{n-3}$, where T^*_3 is that in Fig. 1.*

Proof. First consider the case when T_n is strong. Let

$$v_1 \to v_2 \to \dots \to v_n$$

be its Hamiltonian cycle. If $v_i \to v_{i-2}$ for some $i \in \{1,2,\dots,n\}$ (all sums are modulo n), then, obviously, there is $H(n,3)$ in T_n. So

(12) $$v_i \to v_{i+2}$$

for each $i \in \{1,2,\dots,n\}$. Let T_4 be the tournament on vertices v_1, v_2, v_3, v_n. Its Hamiltonian path is, by (12),

$$H_4' : v_1 \to v_3 \to v_n \to v_2$$

or

$$H_4'' : v_1 \to v_n \to v_3 \to v_2$$

according to $v_3 \to v_n$ or $v_n \to v_3$.

On the other hand, the subtournament $T_n \setminus T_4$ has, by (12), a Hamiltonian path $v_{n-1} \to v_{n-2} \to \dots \to v_5 \to v_4$, where $v_i \to v_{i+2}$ for each $i \in \{4,5,\dots,n-3\}$. According to the lemma, there is a Hamiltonian path H_{n-4} of $T_n - T_4$ starting at v_4. Connectin-ing H_4' or H_4'' and H_{n-4} (it is possible since $v_2 \to v_4$), we obtain $H(n,3)$.

Now, assume that T_n is not strong. Let $T_n^{(1)} \to T_n^{(2)} \to \to \dots$ be its decomposition into strong components. If $|V(T_n^1)| \le \le 3$, the assertion follows immediately by (A). If $|V(T_n^1)| = = n_1 \ge 4$, then there is $H(n_1,3)$ in $T_n^{(1)}$ ($T_n^{(1)}$ is strong) which, with any Hamiltonian path of $T_n - T_4$, forms $H(n,3)$ in T_n.

Theorem 3. $H(n,n-1)$ *is* n-*unavoidable for each* n ($n \ge 3$), *unless* $n = 4$ *and* $T_4 = T_3^* \to v$ *and* $n = 6$ *and* $T_6 = T_5^* \to \to v$, *where* T_3^* *and* T_5^* *are those in Fig. 1.*

Proof. If $n \le 5$ and T_n is none of the forbidden types, it is easy to verify the theorem.

Suppose that for $n \ge 6$ there exists a tournament T_n, on n vertices, without $H(n,n-1)$. We shall show that it produces a contradiction.

Let $\{v_1,v_2,\dots,v_{n-1}\}$ be the vertex set of the tournament T_n. According to the conditions of theorem and (B), there is $H(n-1,n-1)$ - a Hamiltonian bypass in $T_n \to v$. We can assume it is

$$v_1 \to v_2 \to \dots \to v_{n-1}$$

where

$$v_1 \to v_{n-1}. \tag{13}$$

Then, obviously,

$$v \to v_{n-1}. \tag{14}$$

Case 1. $v \to v_1$. Since $v \to v_1 \to \dots \to v_{n-2} \to v_{n-1}$ would be $H(n,n-1)$ if $v \to v_{n-2}$, it follows that

$$v_{n-2} \to v. \tag{15}$$

Two subcases are characteristic.

Subcase 1.1. v_{n-1} dominates none of the v_i, $i \in \{2, 3, \dots, n-2\}$. Then, T_n is of the type $T_{n-1} \to v_{n-1}$ (by (13) and (14)) and a Hamiltonian bypass - $H(n-1,n-1)$ of T_{n-1} joined to v_{n-1} forms $H(n,n-1)$ in T_n.

Subcase 1.2. v_{n-1} dominates at least one of the v_i, $i \in \{2,3,\dots,n-2\}$. In that case, v_{n-1} can be inserted in the path $v_1 \to v_2 \to \dots \to v_{n-2}$. By (15), it produces $H(n,n-1)$ - $v_1 \to \dots \to v_{n-1} \to \dots \to v_{n-2} \to v$, if $v_1 \to v_{n-2}$. So

$$v_{n-2} \to v_1. \tag{16}$$

Similarly, $H(n,n-1)$ - $v \to v_1 \to \dots \to v_{n-1} \to \dots \to v_{n-3} \to v_{n-2}$ implies

$$v_{n-3} \to v. \tag{17}$$

Now, from (14), (15), (16) and (17), it follows that

$$v_{n-2} \to v_1 \to v_2 \to \dots \to v_{n-3} \to v \to v_{n-1}$$

$H(n,n-1)$ in T_n.

Case 2. $v_1 \to v$. Then,

$$v \to v_{n-2}, \tag{18}$$

since $v_1 \to v_2 \to \dots \to v_{n-2} \to v \to v_{n-1}$ is $H(n,n-1)$ in T_n.

By (18), v can be inserted in the path $v_1 \to v_2 \to \dots \to v_{n-2}$ and $H(n,n-1)$ implies

$$v_{n-2} \to v_1. \tag{19}$$

On the other hand, $H(n,n-1) - v \to v_{n-1} \to v_2 \to v_3 \to \dots \to v_{n-2} \to$ $\to v_1$, which follows from (14) and (19), forces

(20) $$v_2 \to v_{n-1}.$$

Now we shall consider two possibilities.

Subcase 2.1. $\{v_3, v_4, \dots, v_{n-3}\} \to v_{n-1}$. Then T_n is of the type $T_{n-1} \to v_{n-1}$, and it is the subcase 1.1.

Subcase 2.2. v_{n-1} dominates at least one of vertices v_i, $i \in \{3,4,\dots,n-1\}$. Then, v_{n-1} can be inserted in the path $v_2 \to$ $\to v_3 \to \dots \to v_{n-2}$ (by (16)), and $H(n,n-1) - v \to v_2 \to \dots \to v_{n-1}$ $\to \dots \to v_{n-2} \to v_1$ induces

(21) $$v_2 \to v.$$

But now we get, by (14), (16) and (21), H(n,n-1)

$$v_2 \to v_3 \to \dots \to v_{n-2} \to v_1 \to v \to v_{n-1},$$

proving the theorem.

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REZIME

O NEIZBEŽNIM PODDIGRAFOVIMA TURNIRA

U radu se pokazuje da se digrafovi H(n,3), H(n,4) i H(n,n-1) pojavljuju, sem u nekoliko izuzetaka, u svakom turniru. Time se delimično potvrdjuje hipoteza V. Sósa.

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