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DEGREE FREQUENCIES IN 3-PARTITE TOURNAMENTS

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ABSTRACT

It is proved that each non-empty set of positive integers is the frequency set of a 3-partite tournament and such a tournament with minimal possible number of vertices is determined.

The number of vertices of a digraph D having a particular outdegre d (indegree d) is the *frequency* of the outdegree (indegree). The set of distinct frequencies of outdegrees appearing in D is the *frequency set of outdegrees* - F^+: The *frequency set of indegrees*, F^-, is defined similarly. If $F^+ = F^-$, this set is called the *frequency set* of D.

A *k-partite tournament* $T(X_1,X_2,\ldots,X_k)$ is a digraph whose vertex set V(T) is the union of k disjoint non-empty sets, partition sets, $X_1,X_2,\ldots,X_k$ and whose arc set contains exactly one of the arcs $\overrightarrow{x_ix_j}$ and $\overrightarrow{x_jx_i}$ for each $x_i \in X_i$, each $x_j \in X_j$ and each $\{i,j\} \subset \{1,2,\ldots,k\}$. $A \to B$ denotes that every vertex of A dominates every vertex of B, where A and B are any two disjoint subsets of V(T).

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Let $f_1, f_2, \ldots, f_n$ $(0 < f_1 < f_2 < \ldots < f_n)$ be a non-empty set of positive integers, then $N_k(f_1, f_2, \ldots, f_n)$ is defined as the smallest possible number such that there exists a k-partite tournament on N_k vertices whose frequency set is $F = \{f_1, f_2, \ldots, f_n\}$. As it was noted in [1] and [3],

$$N_k(f_1, f_2, \ldots, f_n) \geq \sum_{i=1}^{n} f_i$$

clearly holds.

The questions concerning N_1 and N_2 have been treated in [1] and [3]. We shall present the corresponding result for N_3.

The particular case $n = 1$ is covered by the following lemma.

Lemma. *Let* f *be a positive integer. Then there exists a 3-partite tournament whose frequency set is* $\{f\}$ *and* $N_3(f) = 3f$ *unless either* .

(a) $f \equiv 0 \pmod 3$, *in which case* $N_3(f) = f$,

or

(b) $f \not\equiv 0 \pmod 3$ *and* $f \equiv 0 \pmod 4$ *in which case* $N_3(f) = 2f$.

Proof. Since the 3-partite tournament $T_1(X_1, X_2, X_3)$ given by $|X_1| = |X_2| = |X_3| = f$ and $X_1 \to (X_2 \cup X_3)$, $X_2 \to X_3$ (all the others arcs are directed from X_i to X_j for each $i > j$) has $\{f\}$ as its frequency set then

$$N_3(f) \leq 3f.$$

(This notation will be used throught the paper.)
If $f = 3k$ then the tournament $T_1(X_1, X_2, X_3)$ defined as: $|X_1| = |X_2| = |X_3| = k$ and $X_1 \to X_2$, $X_2 \to X_3$ establishes (a).

Suppose $f \not\equiv 0 \pmod 3$ and $N_3(f) = f$. Let $T_3(X_1, X_2, X_3)$ be a 3-partite tournament on f vertices whose frequency set is $\{f\}$. This means that all vertices of T have the same out degree, say a, and the same indegree, say b. Then

$$b = |X_2| + |X_3| - a = |X_3| + |X_1| - a = \\ = |X_1| + |X_2| - a$$

obviously holds and implies $|X_1| = |X_2| = |X_3|$, i.e. $|V(T)| = f \equiv 0 \pmod 3$, which contradicts the assumption. Therefore,

$$N_3(f) \geq 2f \quad \text{for} \quad f \not\equiv 0 \pmod 3.$$

If $f = 4k$, then the 3-partite tournament $T_4(X_1, X_2, X_3)$, given by

$$X_1 = A_1 \cup A_2$$
$$X_2 = A_3 \cup A_4$$
$$X_3 = A_5$$

$|A_i| = k$ $(i = 1,2,3,4)$, $|A_5| = 4k$, $A_1 \to A_3$, $A_2 \to A_4$ has $\{f\}$ as its frequency set. Since $|V(T)| = 8k = 2f$, assertion (b) is proved.

So assume that $f \not\equiv 0 \pmod 3$, $f \not\equiv 0 \pmod 4$ and that there exists a 3-partite tournament $T(X,Y,Z)$ on $2f$ vertices whose frequency set is $\{f\}$. We shall show that it leads to a contradiction.

Let a and b $(a > b)$ be two distinct outdegrees occurring in T with frequencies f and X_1, X_2 vertices of X having outdegrees a and b, respectively. The subsets Y_1, Y_2 and Z_1, Z_2 of Y and Z are defined similarly. Let $|X| = x$, $|Y| = y$ and $|Z| = z$. Then

$$x + y + z = 2f = |V(T)|. \tag{1}$$

We shall consider the following particular cases.

Case 1. *None of the sets* $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ *is empty.*

Then the indegrees occuring in T are

$$d_1^- = y + z - a \qquad d_2^- = y + z - b$$

$$d_3^- = z + x - a \qquad d_4^- = z + x - b$$

$$d_5^- = x + y - a \qquad d_6^- = x + y - b\ .$$

Since there are only two distinct values among d_i^-s $(i = 1,2,3,4,5,6)$, and since $d_i^- \neq d_{i+1}^-$ $(i = 1,2,3,4,5)$, we may assume w.l.g. that $d_1^- = d_3^-$, $d_2^- = d_4^-$ and $x = y$. Now we have

$$d_1^- = d_3^- = x + z - a \qquad d_2^- = d_4^- = x + z - b$$

$$d_5^- = 2x - a \qquad d_6^- = 2x - b\ .$$

Applying the same reason, we get $d_1^- = d_3^- = d_5^-$ and $d_2^- = d_4^- = d_6^-$, which gives $x = y = z$, contradicting by (1) the assumption $f \not\equiv 0 (\mathrm{mod}\ 3)$ or $d_1^- = d_3^- = d_6^-$ and $d_2^- = d_4^- = d_5^-$, which implies $a = b$, contradicting the fact $a > b$.

Case 2. *Exactly one of the sets* X_1, X_2, Y_1, Y_2, Z_1, Z_2 *is empty.*
We may assume, by symmetry, that it is X_2. In that case we have

$$d_1^- = y + z - a$$

$$d_3^- = z + x - a \qquad d_5^- = z + x - b$$

$$d_5^- = x + y - a \qquad d_6^- = x + y - b$$

Following the aforementioned reason, we get

(a) $d_1^- = d_3^-$. It follows that $x = y$ and

$$d_1^- = d_3^- = x + z - a \qquad d_4^- = x + z - b$$

$$d_5^- = 2x - a \qquad d_6^- = 2x - b.$$

This gives $d_1^- = d_3^- = d_5^-$, which implies $x = y = z$ or $d_1^- = d_3^- = d_6^-$ and $d_4^- = d_5^-$, which implies $a = b$; in both cases, a contradiction.

(b) $d_1^- = d_5^-$. Similar to (a).

(c) $d_3^- = d_5^-$. Then $y = z$ and

$$d_1^- = 2y - a$$

$$d_3^- = d_5^- = x + y - a \qquad d_4^- = d_6^- = x + y - b.$$

If $d_1^- = d_3^- = d_5^-$, then $x = y = z$, a contradiction. So $d_1^- = d_4^- = d_6^-$. This gives $y = x + a - b$ and

$$d_3^- = d_5^- = 2x - a \qquad d_1^- = d_4^- = d_6^- = 2x + a - 2b.$$

If $E(T)$ is the arc set of T, then the equality

$$(2) \qquad f(a+b) = f((2x-a) + (2x+a-2b)) = |E(T)|$$

clearly holds, and we get $x = b$, $y = z = a$.

From an obvious equality

$$(3) \qquad E(T) = xy + yz + zx,$$

and the fact $f = (x + y + z)/2 = (2a + b)/2$, it follows that

$$(2a+b)(a+b)/2 = a \quad + 2ab$$

and $a = b$, a contradiction. Case 2 is settled.

Case 3. *Exactly two of the sets* X_1, X_2, Y_1, Y_2, Z_1, Z_2 *are empty.*
(Note that they cannot be X_1 and X_2 and similarly Y_1, Y_2 and Z_1, Z_2.) There are two essentialy different subcases.

Subcase 3.1. $X_2 = Y_2 = \emptyset$.

Then

$$d_1^- = y + z - a$$

$$d_3^- = z + x - a$$

$$d_5^- = x + y - a \qquad d_6^- = x + y - b.$$

Now we have

(a) $d_1^- = d_3^-$. It follows that $x = y$ and $d_1^- = d_3^- = d_6^- = 2x - b$ (because $d_1^- = d_3^- = d_5^-$ leads to $x = y = z$). This gives $x = y = (a + b)/2$, $z = (3a - b)/$ and $f = (x + y + z)/2 = (5a + b)/4$. Now, using (2) and (3) we get

$$(5a + b)(a + b)/4 = (a + b)^2/4 + 2(a + b)(3a - b)/4$$

which gives $a = b$.

(b) $d_1^- = d_5^-$. As (a).

(c) $d_3^- = d_5^-$. As (a).

Subcase 3.2. $X_2 = Z_1 = \emptyset$.

Then

$$d_1^- = y + z - a$$

$$d_3^- = z + x - a \qquad d_4^- = z + x - b$$

$$d_6^- = x + y - b$$

(a) $d_1^- = d_3^- = d_6^-$. It implies that $x = y$, $z = x + a - b$ and we have 3.1.(a).

(b) $d_1^- = d_4^- = d_6^-$. It implies that $y = z$, $y = x + a - b$, and it is again 3.1.(a).

(c) $d_1^- = d_4^-$ and $d_3^- = d_6^-$. Then $y = x + a - b$ and $z = x + 2a - 2b$. So,

$$d_1^- = d_4^- = 2x + 2a - b$$

$$d_3^- = d_6^- = 2x + a - 2b$$

Using (2), we get $x = (-a + 3b)/2$, $y = (a + b)/2$, $z = (3a - b)/2$, $f = 3(a + b)/4$. Substituting in (3), we obtain

$$3(a + b)(a + b)/4 = (-a + 3b)(a + b)/4 + (a + b)(3a - b)/4 + (3a - b)(-a + 3b)/4$$

and $a = b$.

Case 4. *Exactly three of the sets* $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ *are empty*. (Note that the case $X_i = Y_i = Z_i = 0$ is impossible). Assume that $X_2 = Y_2 = Z_1 = 0$. Now we have

$$d_1^- = y + z - a$$

$$d_3^- = z + x - a$$

$$d_6^- = x + y - b.$$

It is clear that

$$f = x + y = z$$

(a) $d_1^- = d_3^-$. Then $x = y$, $f = z = 2x$ and

$$d_1^- = d_3^- = 3x - a \qquad d_6^- = 2x - b.$$

By (2), it follows that $5x = 2(a + b)$, and clearly $x = 2k$. But thus implies $f = 2x = 4k$ which contradicts the assumption $f \not\equiv 0 \pmod 4$.

(b) $d_3^- = d_6^-$. Then $z = y + a - b$, $x = a - b$ and

$$d_1^- = 2y - b \qquad d_3^- = d_6^- = y + a - 2b.$$

It follows, by (2), that $y = 4b/3$ and $f = z = (3a + b)/3$. Now it gives

$$(3a + b)/3 = (a - b)4b/3 + 4b(3a + b)/3 + (3a + b)(a - b)/3$$

or $b(7b - 3a) = 0$. Since $b \neq 0$ (because $y = 4b/3$ and $y \neq 0$), then $3a = 7b$. It implies $f = 8b/3$, and, therefore, $f \equiv 0 (\mathrm{mod}\ 4)$. This contradiction completes the proof of the lemma.

Now, we shall prove the main theorem.

Theorem. *Let* $F = \{f_1, f_2, \ldots, f_n\}$, $(n > 1)$, $0 < f_1 < f_2 < \ldots < f_n$, *be any nonempty set of positive integers. Then there exists a 3-partite tournament whose frequency set is and*

$$N_3(f_1, f_2, \ldots, f_n) = \sum_{i=1}^{n} f_i$$

unless

$n = 2$ *and* $f_1 = 1$, $f_2 = 2$ *in which case* $N_3(1,2) = 4$.

Proof.

Case 1. $n = 2k + 1$ $(k \geq 1)$. A 3-partite tournament $T_1 = T(X_1, X_2, X_3)$, which establishes (1), can be constructed as follows. The partition sets are

$$X_1 = A_1 \cup A_2 \cup \ldots \cup A_{k-1} \cup A_{2k}$$
$$X_2 = A_k \cup A_{k+1} \cup \ldots \cup A_{2k-1}$$
$$X_3 = A_{2k+1},$$

where $|A_i| = f_i$ $(i = 1, 2, \ldots, 2k+1)$, $A_i \cap A_j = \emptyset$ $(i \neq j)$ and the arc set is given by

$$A_i \rightarrow A_{2k+1-i} \quad \text{for } i = 1, 2, \ldots, k-1.$$
$$X_2 \rightarrow X_3$$

Obviously,

$$|V(T_1)| = \sum_{i=1}^{2k+1} f_i$$

and all vertices belonging to a particular subset A_i have the same outdegree (indegree) in T . Denote the outdegree (indegree) by d_i^+ (d_i^-) and denote by S_1 and S_2 the sums

$$S_1 = f_1 + f_2 + \dots + f_{k-1} + f_{2k} = |X_1|$$

$$S_2 = f_k + f_{k+1} + \dots + f_{2k-1} = |X_2|$$

From the definition of T_1, we obtain

$$d_i^+ = f_{2k-1-i} \quad \text{for} \quad i = 1,2,\dots,k-1,$$

$$d_j^+ = S_1 + f_{2k+1} - f_{2k-1-j} \quad \text{for} \quad j = k,k+1,\dots,2k-2,$$

$$d_{2k-1}^+ = S_1 + f_{2k+1}$$

$$d_{2k}^+ = 0$$

$$d_{2k+1}^+ = S_1$$

and

$$d_i^- = S_2 + f_{2k+1} - f_{2k-1-i} \quad \text{for} \quad i = 1,2,\dots,k-1,$$

$$d_j^- = f_{2k-1-j} \quad \text{for } j = k,k+1,\dots,2k-2,$$

$$d_{2k-1}^- = 0$$

$$d_{2k}^- = S_2 + f_{2k+1}$$

$$d_{2k+1}^- = S_2.$$

Since $0 < f_1 < f_2 < \dots < f_{2k+1}$, all d_i^+s (d_i^-) ($i = 1,2,\dots,2k+1$) are distinct. This implies that the frequency set of T_1 is $\{f_1,f_2,\dots,f_{2k+1}\}$.

Case 2. $n = 2k+2$ ($k \geq 1$).

Consider the 3-partite tournament $T_2 = T(X_1,X_2,X_3)$ defined by

$$X_1 = A_1 \cup A_2 \cup \dots \cup A_k \cup A_{2k+1}$$
$$X_2 = A_{k+1} \cup A_{k+2} \cup \dots \cup A_{2k}$$
$$X_3 = A_{2k+2},$$

where $|A_i| = f_i$ $(i = 1,2,\dots,2k+2)$, $A_i \cap A_j = \emptyset$ $(i \neq j)$

$$A_i \to A_{2k+1-i} \quad \text{for} \quad i = 1,2,\dots,k$$
$$X_2 \to X_3.$$

Using the former notation and putting

$$S_3 = f_1 + f_2 + \dots + f_k + f_{2k+1} = |X_1|$$
$$S_4 = f_{k+1} + f_{k+2} + \dots + f_{2k} = |X_2|,$$

we get

$$d_i^+ = f_{2k+1-i} \quad \text{for} \quad i = 1,2,\dots,k$$
$$d_j^+ = S_3 + f_{2k+2} - f_{2k+1-j} \quad \text{for } j = k+1,k+2,\dots,2k$$
$$d_{2k+1}^+ = 0$$
$$d_{2k+2}^+ = S_3$$

and

$$d_i^- = S_4 + f_{2k+2} - f_{2k+1-i}, \quad \text{for} \quad i = 1,2,\dots,k$$
$$d_j^- = f_{2k+1-j} \quad \text{for } j = k+1,k+2,\dots,2k$$
$$d_{2k+1}^- = S_4 + f_{2k+2}$$
$$d_{2k+2}^- = S_4.$$

As in the Case 1, we conclude that the frequency set of T_2 is $\{f_1,f_2,\dots,f_{2k+2}\}$ and

$$|V(T_2)| = \sum_{i=1}^{2k+2} f_i .$$

So, the theorem is proved for $n \geq 3$.

Case 3. $n = 2$.

Let $F = \{f_1, f_2\}$, where $\{f_1, f_2\} \neq \{1,2\}$. If $f_2 \neq 2f_1$, the 3-partite tournament $T_3 = T(X_1, X_2, X_3)$ given by $|X_1| = |X_2| = f_1$, $|X_3| = f_2 - f_1 \neq f_1$, $X_1 \to X_2 \to X_3$, satisfies (1).

If $f_2 = 2f_1$, we distinct two subcases:

Subcase 1. $f_1 = 2k$ $(k \geq 1)$.

Then we construct the tournament $T_4 = T(X_1, X_2, X_3)$ according to

$$X_1 = A_1 \cup A_2$$

$$X_2 = A_3 \cup A_4$$

$$X_3 = A_5 .$$

$|A_i| = k$ $(i = 1,2,3,4)$, $|A_5| = 2k$, $A_i \cap A_j = \emptyset$ $(i \neq j)$, $A_1 \to A_3$, $A_2 \to A_4$, $X_1 \to X_3$, $X_2 \to X_3$. It is easy to see that distinct outdegrees (indegrees) occurring in T_4 are 0 and $3k$ ($4k$ and k) with frequencies f_1 and f_2, respectively, and that $|V(T_4)| = 6k = f_1 + f_2$.

Subcase 2. $f_1 = 2k+1$ $(k \geq 1)$.

Let $T_5 (= T(X_1, X_2, X_3))$ be the 3-partite tournament whose partite sets are

$$X_1 = \{u_1, u_2, \ldots, u_{2k+1}\}$$

$$X_2 = \{v_1, v_2, \ldots, v_{2k+1}\}$$

$$X_3 = \{w_1, w_2, \ldots, w_{2k+1}\},$$

and whose arc set is given by

$$u_i \to v_i \quad \text{for} \quad i = 1, 2, \ldots, 2k+1$$

$$v_j \to \{w_{(j-1)(k+1)+1}, w_{(j-1)(k+1)+2}, \dots, w_{j(k+1)}\}$$

for $j = 1,2,\dots,2k+1$. All vertices of X_1 have outdegree 1 and indegree $4k+1$, while all vertices of X_2 and X_3 have outdegree $3k+1$ and indegree $k+1$. Thus $|V(T_8)| = 6k+3 = f_1 + f_2$ and the frequency set of T_8 is $\{f_1,f_2\}$.

For $F = \{1,2\}$, there is no 3-partite tournament on vertices whose frequency set is F. Indeed, such a tournament on 3 vertices has a frequency set $\{1\}$ or $\{3\}$. Thus, $N_3(1,2) \geq$ ≥ 4. The 3-partite tournament $T_7 = T(X_1,X_2,X_3)$ defined by

$$|X_1| = |X_2| = 1, \quad |X_3| = 2, \quad X_1 \to (X_2 \cup X_4)$$

has the frequency set $\{1,2\}$. This implies that $N_3(1,2) = 4$.

The theorem is proved.

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REZIME

FREKVENCIJE STEPENA ČVOROVA U TRIPARTITNIM TURNIRIMA

U ovom radu pokazano je da je svaki neprazan skup prirodnih brojeva skup frekvencija izlaznih i ulaznih stepena čvorova nekog tripartitnog turnira i pritom su odredjeni turniri sa minimalnim mogućim brojem čvorova.

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