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ON THE LATTICE OF $\mathcal{L}$-VALUED SUBALGEBRAS OF AN ALGEBRA

G. Vojvodić and B. Šešelja

*Institute of Mathematics University of Novi Sad,
Dr Ilije Djuričića 4, 21000 Novi Sad, Yugoslavia*

ABSTRACT

The notion of an $\mathcal{L}$-valued (i.e. fuzzy) algebraic closure system over a set is defined, where $\mathcal{L}$ is a complete lattice. If $\mathcal{L}$ is algebraic, then an $\mathcal{L}$-valued algebraic closure system determines an algebraic lattice.

For a given algebra $\mathcal{A} = (A,F)$, the set $\overline{S_{\mathcal{L}}(\mathcal{A})}$ of its $\mathcal{L}$-valued subalgebras is an $\mathcal{L}$-valued algebraic closure system over A (and thus $(\overline{S_{\mathcal{L}}(\mathcal{A})},\subseteq)$, is an algebraic lattice), if $\mathcal{L}$ is complete, and consists of compact elements only.

1. Let $A \neq \emptyset$, and let $\mathcal{L} = (L,\wedge,\vee,0,1)$ be a complete lattice. Let $\bar{\mathcal{A}} \subseteq \overline{\mathcal{P}(A)}$, i.e. $\bar{\mathcal{A}} = \{\bar{A}_i : \bar{A}_i : A \to L,\ i \in I\}$ is a family of $\mathcal{L}$-valued sets on A (i.e. fuzzy sets on A). Then $\bar{\mathcal{A}}$ is an *$\mathcal{L}$-valued closure system over* A, if $\bar{\mathcal{A}}$ is closed under the arbitrary intersections (note that the intersection and the union of $\mathcal{L}$-valued sets are defined by means of the lattice operations: If $\{\bar{A}_j \mid j \in J\} \subseteq \bar{\mathcal{A}}$, then

$$\bigcap_{j\in J} \bar{A}_j,\quad \bigcup_{j\in J} \bar{A}_j : A \to L,$$

and

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$$\bigcap_{j\in J} \bar{A}_j(a) = \bigwedge_{j\in J} \bar{A}_j(a), \quad \bigcup_{j\in J} \bar{A}_j(a) = \bigvee_{j\in J} \bar{A}_j(a),$$

for every $a \in A$). As usual, we set $A \in \bar{\mathcal{A}}$ (A is here identified with its characteristic function: for every $a \in A$, $A(a) = 1 \in L$).

If $\bar{X} : A \to L$ is an arbitrary $\mathcal{L}$-valued set on A, then

$$[\bar{X}] \overset{def}{=} \bigcap (\bar{B} \mid \bar{B} \in \bar{\mathcal{A}}, \text{ and } \bar{X} \subseteq \bar{B}).$$

We say that $\bar{X} : A \to L$ is *generated by* $\bar{X}$ *in* $\bar{\mathcal{A}}$.

It is obvious that:

(a) $[\bar{X}]$ exists for every $\bar{X} : A \to L$;

(b) $[\bar{X}] \in \bar{\mathcal{A}}$;

(c) $[\bar{X}]$ is the smallest (in the sense of $\subseteq$) in $\bar{\mathcal{A}}$, containing $\bar{X}$;

(d) If $\bar{X} \subseteq \bar{Y}$, then $[\bar{X}] \subseteq [\bar{Y}]$;

(e) $[[\bar{X}]] = [\bar{X}]$.

Lemma 1.1. *If* $\bar{\mathcal{A}}$ *is an* $\mathcal{L}$*-valued closure system over A, then* $(\bar{\mathcal{A}}, \subseteq)$ *is a complete lattice.*

Proof. $\bar{\mathcal{A}}$ contains the greatest element (A), and it is closed under the arbitrary intersections. □

The algebraic $\mathcal{L}$*-valued closure system over* A ($\mathcal{L}$-ACS over A) is a family $\bar{\mathcal{A}} = \{\bar{A}_i \mid \bar{A}_i : A \to L, i \in I\} \subseteq \overline{\mathcal{P}(A)}$, such that:

(I) $\bar{\mathcal{A}}$ is an $\mathcal{L}$-valued closure system over A, and

(II) If $\emptyset \neq \bar{\mathcal{B}} \subseteq \bar{\mathcal{A}}$, and $\bar{\mathcal{B}}$ is a directed family in the sense of $\subseteq$ (that is, for every twoelement (and thus for every finite) subset of $\bar{\mathcal{B}}$ there is an upper bound in $\bar{\mathcal{B}}$), then

$$\bigcup_{\bar{B}_k \in \bar{\mathbf{B}}} \bar{B}_k \in \bar{\mathbf{A}}.$$

In the following, $\bar{\mathbf{A}}$ is an $\mathcal{L}$-ACS over A.

The proofs of the following two lemmas are straightforward (they are similar to the proofs of the corresponding propositions in [4]).

Lemma 1.2. *Let* $\bar{X} : A \to L$. *Then*

$$[\bar{X}] = \cup([\bar{Y}] \mid \bar{Y} \subseteq \bar{X} \text{ and } \{a \in A \mid \bar{Y}(a) > 0\} \text{ is finite}).\square$$

Lemma 1.3. *In the lattice* $(\bar{\mathbf{A}},\subseteq)$,

$$\vee(\bar{A}_i \mid i \in I) = [\cup(\bar{A}_i \mid i \in I)].\square$$

Lemma 1.4. *Let* $\bar{\mathbf{A}}$ *be an* $\mathcal{L}$-ACS *over* A, *where* $\mathcal{L}$ *is an algebraic lattice. Then* $\bar{\mathbf{B}} \in \bar{\mathbf{A}}$ *is compact in* $(\bar{\mathbf{A}},\subseteq)$ *iff* $\bar{B} = [\bar{X}]$, *for some* $\bar{X} : A \to L$, *such that* $\{a \mid \bar{X}(a) > 0\}$ *is finite.*

Proof. Let $\bar{B} = [\bar{X}]$, $\bar{X} : A \to L$, and $\{a \mid \bar{X}(a) > 0\}$ is finite. If $\bar{B} \subseteq \vee(\bar{A}_i \mid i \in I)$, then by Lemma 1.3.,

$$\bar{X} \subseteq [\bar{X}] = \bar{B} \subseteq \bigvee_{i\in I} \bar{A}_i = [\bigcup_{i\in I} \bar{A}_i].$$

Let now for $j \in \{1,\dots,n\}$, $\bar{X}(a_j) > 0$, and for every $a \in A \setminus \{a_j \mid j \in \{1,\dots,n\}\}$, let $\bar{X}(a) = 0$. Then,

$$[\bigcup_{i\in I} \bar{A}_i](a_j) > 0, \text{ for } j \in \{1,\dots,n\},$$

and by Lemma 1.2., there is $\bar{H}_j \subseteq \bigcup_{i\in I} \bar{A}_i$, such that $\{a \mid \bar{H}_j(a) > 0\}$ is finite, and $[\bar{H}_j](a_j) > 0$. Since $\mathcal{L}$ is algebraic, and by virtue of the inequality

$$\bar{H}_j(a) \le \bigvee_{i\in I} \bar{A}_i(a),$$

it follows that

$$\bar{H}_j(a) \leq \bigvee_{i=1}^{m_j} \bar{A}_i(a).$$

Since there is only a finite number of $a_{j_k} \in A$ such that $\bar{H}_j(a_{j_k}) > 0$, it follows that

$$\bar{H}_j \subseteq \bigcup_{i=1}^{M_j} \bar{A}_i, \text{ where } M_j = m_{j_1} + \ldots + m_{j_n}.$$

Let $M = \bigcup_j M_j$. Then

$$\bar{X} \subseteq [\bigcup_{i=1}^{M} \bar{A}_i],$$

and hence

$$\bar{B} = [\bar{X}] \subseteq [[\bigcup_{i=1}^{M} \bar{A}_i]] = [\bigcup_{i=1}^{M} \bar{A}_i] = \bigvee_{i=1}^{M} \bar{A}_i.$$

Let now $\bar{B} \in \bar{\mathcal{A}}$ be a compact element in $(\bar{\mathcal{A}}, \subseteq)$. Then by Lemma 1.2.,and Lemma 1.3.,

$$\bar{B} = [\bar{B}] = \bigcup_{i\in I} (\bar{Y}_i | \bar{Y}_i \subseteq \bar{B}, \text{ and } \{a | \bar{Y}_i(a) > 0\} \text{ is finite}) =$$

$$= \bigvee_{i\in I} (\bar{Y}_i | \bar{Y}_i \subseteq \bar{B}, \text{ and } \{a | \bar{Y}_i(a) > 0\} \text{ is finite}) =$$

$$= \bigvee_{i=1}^{k} (\bar{Y}_i | \bar{Y}_i \subseteq \bar{B}, \text{ and } \{a | \bar{Y}_i(a) > 0\} \text{ is finite}),$$

since $\bar{B}$ is compact. Let $\bar{Y} = \bigcup_{i=1}^{k} \bar{Y}_i$. Obviously, $\bar{Y} \subseteq \bar{B}$, and $\{a | \bar{Y}(a) > 0\}$ is finite. Hence, by Lemma 1.3.,

$$\bar{B} = [\bar{Y}_1] \vee \ldots \vee [\bar{Y}_k] = [\bar{Y}_1 \cup \ldots \cup \bar{Y}_k] = [\bar{Y}]. \ \square$$

Proposition 1.5. *If $\bar{\mathcal{A}}$ is an $\mathcal{L}$-ACS on A, and $\mathcal{L}$ is an algebraic lattice, then $(\bar{\mathcal{A}}, \subseteq)$ is an algebraic lattice as well.*

Proof. By virtue of Lemma 1.2., for every $\bar{X} \in \bar{\mathcal{A}}$,

$$\bar{X} = \cup\ ([\bar{Y}] \mid \bar{Y} \subseteq \bar{X}, \text{ and } \{a \mid \bar{Y}(y) > 0\} \text{ is finite}).$$

By Lemma 1.3.,

$$\bar{X} = \vee\ ([\bar{Y}] \mid \bar{Y} \subseteq \bar{X}\ , \text{ and } \{a \mid \bar{Y}(a) > 0\} \text{ is finite}).$$

Now, by Lemma 1.4., every $[\bar{Y}]$ is compact. □

□

2. Let $\mathbf{A} = (A,F)$ be an algebra, and $K \subseteq A$ a set of its constants (if $K = \emptyset$, we accept the empty set to be a subalgebra of $\mathbf{A}$). An *$\mathcal{L}$-valued* (i.e. fuzzy) *subalgebra of* $\mathbf{A}$ ([2], [3]), where $\mathcal{L}$ is a complete lattice, is any mapping $\bar{B} : A \to L$, such that

(a) $K \subseteq \bar{B}$ (K is identified with its characteristics function), and

(b) $\bar{B}(f(x_1,\dots,x_n)) \geqslant \bar{B}(x_1) \wedge \dots \wedge \bar{B}(x_n)$, for all $x_1,\dots, x_n \in A$, $f \in F_n \subseteq F$, $n \in N$.

We shall denote the set of all $\mathcal{L}$-valued subalgebras of $\mathbf{A}$ by $\overline{S(\mathbf{A})}$.

Proposition 2.1. *Let* $\mathbf{A} = (A,F)$ *be an algebra, and let $\mathcal{L}$ be a complete lattice in which every element is compact. Then,* $\overline{S_{\mathcal{L}}(\mathbf{A})}$ *is an* $\mathcal{L}$-ACS *over* A.

Proof. $\overline{S_{\mathcal{L}}(\mathbf{A})}$ is obviously an $\mathcal{L}$-valued closure system over A. To prove that it is algebraic, consider an arbitrary directed family $\bar{\mathbf{B}} = \{\bar{B}_i \mid i \in I\} \subseteq \overline{S_{\mathcal{L}}(\mathbf{A})}$. If $a \in A$, then the family $\{\bar{B}_i(a) \mid i \in I\}$ is directed in $\mathcal{L}$. Since $\mathcal{L}$ is complete and algebraic, and every element in $\mathcal{L}$ is compact, it follows (see [1]) that every directed family (in $\mathcal{L}$) contains its supremum. Thus,

$$(\bigcup_{i\in I} \bar{B}_i)(a) = \bigvee_{i\in I} \bar{B}_i(a) \in \{\bar{B}_i(a) \mid i \in I\}.$$

It is clear now that for $a_1,\dots,a_n \in A$, there are $\bar{B}_1,\dots,\bar{B}_n \in \bar{\mathcal{B}}$, such that for $j = 1,\dots,n$,

$$\bar{B}_j(a_j) = (\bigcup_{i\in I} \bar{B}_i)(a_j).$$

$\bar{\mathcal{B}}$ is directed, and thus there is $\bar{B} \in \bar{\mathcal{B}}$, such that for $j = 1,\dots,n$,

$$\bar{B}(a_j) = \bigcup_{i\in I} \bar{B}_i(a_j).$$

Hence, since $\bar{B}$ is an $\mathcal{L}$-valued subalgebra of $\mathbf{A}$,

$$\bigwedge_{j=1}^{n} (\bigcup_{i\in I} \bar{B}_i)(a_j) = \bigwedge_{j=1}^{n} \bar{B}(a_j) \le \bar{B}(f(a_j,\dots,a_n)) \le$$

$$\le (\bigcup_{i\in I} \bar{B}_i)(f(a_1,\dots,a_n)),$$

proving that $\bigcup_{i\in I} \bar{B}_i$ belongs to $\overline{S_{\mathcal{L}}(\mathbf{A})}$. □

Corollary 2.2. *Let $\mathcal{L}$ be a complete lattice consisting of compact elements only. Then, for an arbitrary algebra $\mathbf{A}$, the lattice $(\overline{S_{\mathcal{L}}(\mathbf{A})},\subseteq)$ is algebraic.*

Proof. By Proposition 2.1., and Proposition 1.5. □

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REZIME

O MREŽI $\mathcal{L}$-VREDNOSNIH PODALGEBRI DATE ALGEBRE

U radu se definiše pojam $\mathcal{L}$-vrednosnog algebarskog sistema zatvaranja na skupu, gde je $\mathcal{L}$ kompletna mreža. Pokazuje se da $\mathcal{L}$-vrednosni sistem zatvaranja odredjuje algebarsku mrežu, ako je $\mathcal{L}$ algebarska. Razmatra se i mreža $\mathcal{L}$-vrednosnih podalgebri proizvoljne algebre i pokazuje se da je ona algebarska, ako je $\mathcal{L}$ takva, da je njen svaki elemenat kompaktan.

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