

L-VALUED CONGRUENCES AND L-VALUED NORMAL
 SUBGROUPS

G. Vojvodić and B. Šešelja

University of Novi Sad, Faculty of Science,
 Institute of Mathematics, Dr. I. Djuričića 4,
 21000 Novi Sad, Yugoslavia

ABSTRACT

A connection between L-valued (L is a lattice) congruences ([4]) and L-valued normal subgroups of a group is given. It is proved that the corresponding lattices of all such mappings are isomorphic. It is also proved that L is (up to the isomorphism) a sublattice of the lattice of all L-valued normal subgroups of a group, and that the latter is modular (as well as the lattice of all L-valued congruences) if L is infinitely distributive.

1.

Let $(G, \cdot, ^{-1}, e)$ be a group (denoted by G) and $(L, \wedge, \vee, 0, 1)$ a complete lattice (in the following denoted by L).

L-valued congruence relations on G is a mapping $\bar{\rho} : G^2 \rightarrow L$, such that ([4]):

- (r) $\bar{\rho}(x, x) = 1$, for every $x \in G$;
- (s) $\bar{\rho}(x, y) = \bar{\rho}(y, x)$, for all $x, y \in G$;
- (t) $\bar{\rho}(x, y) \geq \bar{\rho}(x, z) \wedge \bar{\rho}(z, y)$, for all $x, y, z \in G$;
- (sub) $\bar{\rho}(x, y) \wedge \bar{\rho}(u, v) \leq \bar{\rho}(x \cdot u, y \cdot v)$, and
 $\bar{\rho}(x, y) \leq \bar{\rho}(x^{-1}, y^{-1})$, for all $x, y, u, v \in G$.

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L -valued subgroup of G ([2]) is a mapping $\bar{H} : G \rightarrow L$, such that:

- (1) $\bar{H}(x \cdot y) \geq \bar{H}(x) \wedge \bar{H}(y)$, for all $x, y \in G$;
- (2) $\bar{H}(x^{-1}) \geq \bar{H}(x)$, for every $x \in G$.
- (3) $\bar{H}(e) = 1$.

The conditions (1), (2) and (3) are equivalent with (3) and (4), where

- (4) $\bar{H}(x \cdot y^{-1}) \geq \bar{H}(x) \wedge \bar{H}(y)$, for all $x, y \in G$ (see [2]).

It is also clear that (1) and (2) imply $\bar{H}(x) = \bar{H}(x^{-1})$ (see [2]).

Let \bar{H} be an L -valued subgroup of G , and let $a \in G$. Define the mappings $a\bar{H}$ and $\bar{H}a$ from G to L in the following way:

$$a\bar{H}(x) \stackrel{\text{def}}{=} \bar{H}(a^{-1} \cdot x), \text{ and } \bar{H}a(x) \stackrel{\text{def}}{=} \bar{H}(x \cdot a^{-1}),$$

for every $x \in G$.

An L -valued subgroup \bar{H} of G is *normal*, if $a\bar{H} = \bar{H}a$, for every $a \in G$ (this fact will be denoted by $\bar{H} \triangleleft G$).

Lemma 1. For every $p \in L$, a p -cut H_p of an L -valued normal subgroup \bar{H} of G is a normal (ordinary) subgroup of G .

Proof. As it is known (see for example [1]), $H_p \subseteq G$, and $x \in H_p$ iff $\bar{H}(x) \geq p$. H_p is a subgroup of G (see [2]). It is normal, since $x \in aH_p$ iff $a\bar{H}(x) \geq p$ iff $\bar{H}a(x) \geq p$ iff $x \in H_p a$. \square

Lemma 2. $a\bar{H} = \bar{H}a$ is equivalent with $\bar{H}(x) \leq \bar{H}(a^{-1} \cdot x \cdot a)$, $x \in G$.

Proof. If $a\bar{H} = \bar{H}a$, then $\bar{H}(a^{-1} \cdot x \cdot a) = a\bar{H}(x \cdot a) =$

$$= \bar{H}a(x; a) = \bar{H}(x \cdot a \cdot a^{-1}) = \bar{H}(x).$$

If $\bar{H}(x) \leq \bar{H}(a^{-1} \cdot x \cdot a)$, then $a\bar{H}(x) = \bar{H}(a^{-1} \cdot x) \leq \bar{H}(a \cdot a^{-1} \cdot x \cdot a^{-1}) = \bar{H}(x \cdot a^{-1}) = \bar{H}a(x)$, and $\bar{H}a(x) = \bar{H}(x \cdot a^{-1}) \leq \bar{H}(a^{-1} \cdot x \cdot a) = \bar{H}(a^{-1} \cdot x) = a\bar{H}(x)$.

Thus, $a\bar{H} = \bar{H}a$. \square

Let $\bar{H} \triangleleft G$, and let $\bar{\rho}_{\bar{H}} : G^2 \rightarrow L$, such that $\bar{\rho}_{\bar{H}}(x, y) \leq \bar{H}(x \cdot y^{-1})$. Then, we have the following proposition:

Proposition 3. $\bar{\rho}_{\bar{H}}$ is an L-valued congruence on G.

Proof.

$$(r): \bar{\rho}_{\bar{H}}(x, x) = \bar{H}(x \cdot x^{-1}) = \bar{H}(e) = 1;$$

$$(s): \bar{\rho}_{\bar{H}}(x, y) = \bar{H}(x \cdot y^{-1}) = \bar{H}((x \cdot y^{-1})^{-1}) = \bar{H}(y \cdot x^{-1}) = \bar{\rho}_{\bar{H}}(y, x);$$

$$(t): \bar{\rho}_{\bar{H}}(x, z) \wedge \bar{\rho}_{\bar{H}}(z, y) = \bar{H}(x \cdot z^{-1}) \wedge \bar{H}(z \cdot y^{-1}) \leq \bar{H}(x \cdot z^{-1} \cdot z \cdot y^{-1}) = \bar{H}(x \cdot y^{-1}) = \bar{\rho}_{\bar{H}}(x, y);$$

$$\begin{aligned} (sub): \bar{\rho}_{\bar{H}}(x \cdot u, y \cdot v) &= \bar{H}(x \cdot u \cdot (y \cdot v)^{-1}) = \bar{H}y \cdot v(x \cdot u) = \\ &= y \cdot v\bar{H}(x \cdot u) = \bar{H}((y \cdot v)^{-1} \cdot x \cdot u) = \\ &= \bar{H}(v^{-1} \cdot y^{-1} \cdot x \cdot u) = v\bar{H}(y^{-1} \cdot x \cdot u) = \\ &= \bar{H}v(y^{-1} \cdot x \cdot u) = \bar{H}(y^{-1} \cdot x \cdot u \cdot v^{-1}) \geq \\ &\geq \bar{H}(y^{-1} \cdot x) \wedge \bar{H}(u \cdot v^{-1}) = \bar{\rho}_{\bar{H}}(x, y) \wedge \bar{\rho}_{\bar{H}}(u, v), \end{aligned}$$

since

$$\bar{H}(y^{-1} \cdot x) = y\bar{H}(x) = \bar{H}y(x) = \bar{H}(x \cdot y^{-1}).$$

For the unary operation we have

$$\begin{aligned} \bar{\rho}_{\bar{H}}(x, y) &= \bar{H}(x \cdot y^{-1}) = \bar{H}((x \cdot y^{-1})^{-1}) = \bar{H}((y^{-1})^{-1} \cdot x^{-1}) = \\ &= y^{-1}\bar{H}(x^{-1}) = \bar{H}y^{-1}(x^{-1}) = \\ &= \bar{H}(x^{-1} \cdot (y^{-1})^{-1}) = \bar{\rho}_{\bar{H}}(x^{-1}, y^{-1}). \square \end{aligned}$$

Let now $\bar{\rho}$ be an L-valued congruence relation on G,

and let $\bar{H}_{\bar{\rho}} : G \rightarrow L$, be such that $\bar{H}_{\bar{\rho}}(x) = \bar{\rho}(e, x)$, $x, e \in G$. Then, we have the following results.

Lemma 4. $\bar{H}_{\bar{\rho}} \triangleleft G$.

Proof. $\bar{H}_{\bar{\rho}}$ is an L-valued subgroup of G, since

$$\begin{aligned}\bar{H}_{\bar{\rho}}(x \cdot y^{-1}) &= \bar{\rho}(e, x \cdot y^{-1}) \geq \bar{\rho}(e, x) \wedge \bar{\rho}(e, y^{-1}) \geq \\ &\geq \bar{\rho}(e, x) \wedge \bar{\rho}(e, y) = \bar{H}_{\bar{\rho}}(x) \wedge \bar{H}_{\bar{\rho}}(y),\end{aligned}$$

and

$$\bar{H}_{\bar{\rho}}(e) = \bar{\rho}(e, e) = 1.$$

$\bar{H}_{\bar{\rho}}$ is normal, since

$$\begin{aligned}\bar{H}_{\bar{\rho}}(a^{-1} \cdot x \cdot a) &= \bar{\rho}(a^{-1} \cdot x \cdot a, e) \geq \bar{\rho}(x \cdot a, a) \wedge \bar{\rho}(a^{-1}, a^{-1}) = \\ &= \bar{\rho}(x \cdot a, a) \geq \bar{\rho}(x, e) = \bar{H}_{\bar{\rho}}(x).\end{aligned}$$

The proof now follows from Lemma 2. \square

Lemma 5. If $\bar{\rho}$ is an L-valued congruence on a group G, and $H \triangleleft G$, then

$$a) \quad \bar{\rho}_{\bar{H}_{\bar{\rho}}} = \bar{\rho}; \quad b) \quad \bar{H}_{\bar{\rho}_{\bar{H}}} = \bar{H}.$$

Proof. Obvious. \square

If " \leq " is the usual inclusion relation on L-valued sets (i.e. $\bar{A} \leq \bar{B}$ iff $\bar{A}(x) \leq \bar{B}(x)$ for every $x \in S$, where $\bar{A}, \bar{B} : S \rightarrow L$), then it is clear that $(\overline{S_{\Delta}(G)}, \leq)$ and $(\overline{C(G)}, \leq)$ are lattices, where $\overline{S_{\Delta}(G)}$ is a set of all L-valued normal subgroups, and $\overline{C(G)}$ a set of all L-valued congruences on G.

Proposition 6. $(\overline{S_{\Delta}(G)}, \leq) \cong (\overline{C(G)}, \leq)$.

Proof. The mapping $\bar{\rho} \mapsto \bar{H}_{\bar{\rho}}$ from $\overline{C(G)}$ to $\overline{S_{\Delta}(G)}$ is a

bijection by Lemma 5, and all we have to prove is that for $\bar{H}, \bar{K} \triangleleft G$

$$\bar{H} \leq \bar{K} \text{ iff } \bar{\rho}_{\bar{H}} \leq \bar{\rho}_{\bar{K}}.$$

If $\bar{H} \leq \bar{K}$, then $\bar{\rho}_{\bar{H}}(x, y) = \bar{H}(x \cdot y^{-1}) \leq \bar{K}(x \cdot y^{-1}) = \bar{\rho}_{\bar{K}}(x, y)$.

If $\bar{\rho}_{\bar{H}} \leq \bar{\rho}_{\bar{K}}$, then $\bar{H}(x) = \bar{\rho}_{\bar{H}}(e, x) \leq \bar{\rho}_{\bar{K}}(e, x) = \bar{K}(x)$. \square

The following result shows "how far" from L the lattice $(\overline{S_{\Delta}(G)}, \leq)$ can go. (In fact, this is a general property of all L-valued algebras).

Proposition 7. *L is (up to the isomorphism) a sublattice of $(\overline{C(G)}, \leq)$.*

Proof. Let $p \in L$, and $\bar{D}_p : G^2 \rightarrow L$, such that

$$\bar{D}_p(x, y) = \begin{cases} 1, & \text{if } x = y \\ p, & \text{if } x \neq y. \end{cases}$$

It is clear that $\bar{D}_p \in \overline{C(G)}$, and that

$$L \cong (\{\bar{D}_p \mid p \in L\}, \leq) \leq (\overline{C(G)}, \leq). \square$$

The properties of $(\overline{S_{\Delta}(G)}, \leq)$ thus depend on the corresponding properties of L. Having this in mind, we shall now investigate the modularity of the former.

Let $\bar{H}, \bar{K} \in \overline{S(G)^{11}}$, and let $\bar{H} \cdot \bar{K} : G \rightarrow L$ be such that

$$\bar{H} \cdot \bar{K}(x) \stackrel{\text{def}}{=} \vee (\bar{H}(y) \wedge \bar{K}(z) \mid y, z \in G, y \cdot z = x).$$

Lemma 8. *If $\bar{H}, \bar{K} \triangleleft G$, then $\bar{H} \cdot \bar{K} \triangleleft G$.*

Proof.

$$\begin{aligned} \bar{H} \cdot \bar{K}(x \cdot y^{-1}) &= \vee (\bar{H}(u) \wedge \bar{K}(v) \mid x \cdot y^{-1} = u \cdot v) \geq \\ &\bar{H}(x) \wedge \bar{K}(y^{-1}) = \bar{H}(x) \wedge \bar{K}(y), \end{aligned}$$

¹¹ $\overline{S(G)}$ is a set of all L-valued subgroups of G.

and $\bar{H} \cdot \bar{K}$ is an L-valued subgroup of G. It is a normal one:

$$\begin{aligned}\bar{H} \cdot \bar{K}(x) &= \vee (\bar{H}(y) \wedge \bar{K}(z) \mid y \cdot z = x) \leq \\ &\leq \vee (\bar{H}(a^{-1} \cdot y \cdot a) \wedge \bar{K}(a^{-1} \cdot z \cdot a) \mid y \cdot z = x) = \\ &= \vee \bar{H}(a^{-1} \cdot y \cdot a) \wedge (\bar{K}(a^{-1} \cdot z \cdot a) \mid a^{-1} y \cdot z \cdot y = a^{-1} \cdot x \cdot a) = \\ &= \bar{H} \cdot \bar{K}(a^{-1} \cdot x \cdot a).\end{aligned}$$

The proof now follows by Lemma 2. \square

Lemma 9. In the lattice $(\overline{S_4(G)}, \leq)$, $\bar{H} \cdot \bar{K} = \bar{H} \vee \bar{K}$.

Proof. We shall show first that for $\bar{H}, \bar{K} \in \overline{S_4(G)}$, $\bar{H}, \bar{K} \leq \bar{H} \cdot \bar{K}$. It follows from the definition of $\bar{H} \cdot \bar{K}$, and since

$$\bar{H}(x) = \bar{H}(x) \wedge \bar{K}(e), \quad \bar{K}(x) = \bar{H}(e) \wedge \bar{K}(x).$$

Now, if $\bar{H}, \bar{K} \leq \bar{M} \in \overline{S(G)}$, then $\bar{H} \cdot \bar{K} \leq \bar{M}$:

$$\begin{aligned}\bar{H} \cdot \bar{K}(x) &= \vee (\bar{H}(y) \wedge \bar{K}(z) \mid y \cdot z = x) \leq \\ &\leq \vee (\bar{M}(y) \wedge \bar{M}(z) \mid y \cdot z = x) \\ &\leq \vee (\bar{M}(y \cdot z) \mid y \cdot z = x) = \bar{M}(x). \quad \square\end{aligned}$$

Proposition 10. Let G be a group, and L a complete distributive (if G is infinite, then infinitely distributive) lattice. Then $(\overline{S(G)}, \leq)$ is a modular lattice.

Proof. We have to prove that from $\bar{H}, \bar{K}, \bar{M} \in \overline{S_4(G)}$, and $\bar{H} \leq \bar{M}$, it follows that $\bar{M} \cap (\bar{K} \cdot \bar{H}) \leq (\bar{M} \cap \bar{K}) \cdot \bar{H}$:

$$\begin{aligned}\bar{M} \cap (\bar{K} \cdot \bar{H})(x) &= \bar{M}(x) \wedge (\vee (\bar{K}(y) \wedge \bar{H}(z) \mid y \cdot z = x)) = \\ &= \vee (\bar{M}(x) \wedge \bar{K}(y) \wedge \bar{H}(z) \mid y \cdot z = x) =\end{aligned}$$

$$= v (\bar{M}(x) \wedge \bar{M}(z^{-1}) \wedge \bar{K}(y) \wedge \bar{H}(z) \mid y \cdot z = x) \leq$$

(since $\bar{H}(z) \leq \bar{M}(z) = \bar{M}(z^{-1})$)

$$\leq v (\bar{M}(x \cdot z^{-1}) \wedge \bar{K}(y) \wedge \bar{H}(z) \mid y \cdot z = x) =$$

$$= v (\bar{M}(y) \wedge \bar{K}(y) \wedge \bar{H}(z) \mid y \cdot z = x) =$$

$$= v ((\bar{M} \cap \bar{K})(y) \wedge \bar{H}(z) \mid y \cdot z = x) = (\bar{M} \cap \bar{K}) \cdot \bar{H}(x). \square$$

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REZIME

L-VREDNOSNE KONGRUENCIJE I L-VREDNOSNE NORMALNE PODGRUPE

Data je veza izmedju L-vrednosnih kongruencija (L je mreža) i L-vrednosnih normalnih podgrupa date grupe. Pokazano je da su odgovarajuće mreže tih preslikavanja izomorfne, da je L uvek podmreža svake od njih, pa tako određuje i njihove osobine. U vezi s tim, dokazano je da je mreža L-vrednosnih normalnih podgrupa date grupe modularna, ako je L beskonačno distributivna mreža.

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