ZBORNIK RADOVA
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Univerziteta u Novom Sadu
Serija za matematiku, 17,1(1987)

REVIEW OF RESEARCH
Faculty of Science
University of Novi Sad
Mathematics Series, 17,1(1987)

THE SPECTRUM OF A NORMAL DIGRAPH

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ABSTRACT

In this paper we consider the spectra of finite digraphs whose 0-1 adjacency matrix is normal but not a symmetric one. Some general properties of such diagraphs are proved, all the normal digraphs whose order is at most 5 are found, and the spectra of such digraphs calculated.

1. INTRODUCTION

In this paper we shall consider (connected and disconnected) digraphs $G = (V,E)$ without multiple edges and without loops. $V = V(G)$ is the set of vertices of G and $E = E(G)$ is the set of its oriented edges, i.e. a set of ordered pairs (x,y) of its vertices $(x \neq y)$. By this definition, all edges of G are obviously simple.

If $x,y \in V(G)$, then $\underline{x}$ is adjacent to $\underline{y}$ means that $(x,y) \in E(G)$. By this definition, for each ordered pair (x,y) of vertices from G $(x \neq y)$ there is at most one arc leading from $\underline{x}$ to $\underline{y}$, and for any pair of distinct vertices x,y there are at most two arcs joining $\underline{x}$ and $\underline{y}$.

AMS Mathematics Subject Classification (1980): 05C20.
Key words and phrases: Normal digraph, spectra.

If x is adjacent to y, and y is nonadjacent to x, we write $x \to y$. If x is adjacent to y and conversely, then we write $x \leftrightarrow y$. If $x \to y$ or if $y \to x$, roughly speaking we say that "edge" xy of G (in fact the corresponding edge of a digraph obtained from G by deleting the orientation of its arcs) is simple. If $x \leftrightarrow y$, we call the "edge" xy double. Hence, any "edge" xy of G is either simple or double.

For a digraph G, let $A = A(G) = [a_{xy}]$ be the 0-1 adjacency matrix of G, where $a_{xy} = 1$ if x is adjacent to y, and $a_{xy} = 0$ otherwise. G is called normal if its adjacency matrix $A = A(G)$ is normal, i.e. if $AA' = A'A$. G is called symmetric if all its edges are double. Such a digraph is obviously normal because its adjacency matrix A(G) is a symmetric one. This case is not interesting for us, because we only want to generalize the spectral theory of symmetric graphs. Hence, we shall search only for non-symmetric (briefly - proper) normal digraphs, which we denote by PND for short.

Let G_o be the simple graph obtained from G by deleting the orientations of arcs in G, then by joining its double edges in simple ones. We call it the basic graph of G, and notice that this graph is obviously unique. We also call G the over-digraph of G_o. The order of G is the order of its basic graph G_o, and is denoted usually by $|G|$. G is called connected if G_o is such.

As the example of the graph $G_o = K_2$ shows, not every graph G_o has a proper normal over-digraph. Hence, in this respect two problems arise:

(i) Find all the graphs G_o having at least one proper normal over-digraph;

(ii) If this class is $\mathcal{G}$ and if $G_o \in \mathcal{G}$, find all proper normal over-digraphs of G_o.

In this paper we shall develop some methods to solve these questions. In particular, we shall prove that some large classes of graphs have only symmetric digraphs as their normal over-digraphs.

If G is a digraph with n vertices, the spectrum $\sigma(G) = \{\lambda_1,\dots,\lambda_n\}$ of G ($\mathrm{Re}\lambda_1 \geq \mathrm{Re}\lambda_n$) is defined to be the spectrum of its adjacency matrix $A = A(G)$ (see [2], p.12). In a general case the spectrum is not real. If G is a proper normal digraph, then its spectrum contains at least one non--real eigenvalue. The spectrum of an arbitrary digraph is obviously symmetric around the real axis.

Because of completeness, in the next theorem we indicate some basic properties of spectra of normal digraphs. Some of them are valid for general digraphs (see [2]), while for instance (vii) and (ix) are specific for normal ones.

THEOREM 1. *Let* $\sigma(G) = \{\lambda_1,\dots,\lambda_n\}$ *be the spectrum of a normal digraph* G ($\mathrm{Re}\lambda_1 \geq \dots \geq \mathrm{Re}\lambda_n$). *Then:*

(i) *Its spectral radius* $r(G) = \lambda_1(G)$ *is real;*

(ii) *All the spectrum* $\sigma(G)$ *lies in the circle* $|\lambda| \leq r(G)$;

(iii) $r(G)$)*is a simple eigenvalue if and only if* G *is a connected digraph;*

(iv) G *is bipartite if and only if* $\sigma(G)$ *is symmetric around the zero;*

(v) *If* G *is connected and* $-r(G) \in \sigma(G)$, *then* G *is bipartite and* $-r(G)$ *is a simple eigenvalue of* G;

(vi) *The spectral trace of* A(G) *is zero, i.e.*

$$\mathrm{tr}(A) = \sum_{j=1}^{n} \lambda_j(G) = 0;$$

(vii) *The numerical range of* $A = A(G)$, *i.e. the set* $W(A) = \{\langle Ax,x\rangle \; \|x\| = 1\}$ *coincides with the convex hull of the spectrum* $\sigma(G)$;

(viii) *There is at least one eigenvector corresponding to* $r(G)$ *whose all coordinates are real and positive;*

(ix) *There is a set of mutually normal eigenvectors* $v_1,\dots,v_n \in H = C^n$, *which correspond respectively to the eigenvalues* $\lambda_1,\dots,\lambda_n$, *which is then an orthonormal basis of the space* H.

If G_1 and G_2 are two digraphs, we say that G_1 is isomorphic to G_2 and we write $G_1 \simeq G_2$ if there is a bijection $\omega: V(G_1) \to V(G_2)$ such that one of the following cases occurs:

1^o) for any two vertices $x,y \in V(G_1)$, $(x,y) \in E(G_1)$ implies that $(\omega(y),\omega(x)) \in E(G_2)$;

2^o) for any two vertices $x,y \in V(G_1)$, $(x,y) \in E(G_1)$ implies $(\omega(y), \omega(x)) \in E(G_2)$.

In the first case we say that G_1 and G_2 have the same orientation, while in the second case - the opposite one. Two isomorphic digraphs obviously have the same order.

If $A_i = A(G_i)$ is the adjacency matrix of the digraph G_i $(i = 1,2)$, then $G_1 = G_2$ if and only if $A_2 = UA_1U^{-1}$ or $A_2 = UA_1'U^{-1}$ for a unitary matrix U. Hence, G_1 is normal if and only if G_2 is such. The spectra of two isomorphic digraphs (including their multiplicities) are obviously same. Hence, with respect to the spectrum, it makes a sense only to search for nonisomorphic normal digraphs.

Throughout this paper we shall use the following equivalent criterion of normality. Let $\underline{x}$ and $\underline{y}$ be any two (not necessarily distinct) vertices of a digraph G. A vertex $z \in V(G)$ is called a <u>common successor</u> (short - suc) of x and y if both x and y are adjacent to z. A vertex $z \in V(G)$ is called a <u>common predecessor</u> (short - prc) of x and y, if z is adjacent to both x and y. Let $suc_G(x,y) = suc(x,y)$ be the number of all common successors of x and y, and $prc_G(x,y) = prc(x,y)$ be the number of all common predecessors of x and y. In particular, $suc(x) = suc(x,x)$ is the number of all the vertices $y \in V(G)$ such that x is adjacent to y, and $prc(x) = prc(x,x)$ is the number of all the vertices $y \in V(G)$ such that y is adjacent to x. We also denote by $sc(x)$ the number of all the vertices $y \in V(G)$ such that $x \to y$, and by $pc(x)$ the number of all the vertices $y \in V(G)$ such that $y \to x$.

Using the definition of normal digraphs, the following proposition is then immediate.

PROPOSITION 1. *A digraph* G *is normal if and only if the following is true:*

(1) $suc(x,y) = prc(x,y)$,

for any two (not necessarily distinct) vertices $x,y \in V(G)$.

In particular, relation (1) means that

(2) $suc(x) = prc(x)$,

or equivalently

(3) $sc(x) = pc(x)$,

for any vertex $x \in V(G)$.

If G is a PND and if $x,y \in V(G)$ $(x \neq y)$, then instead of the sentence "by relation (a) related to the pair (x,y)" we often say for short "by the pair (x,y)", and instead of the sentence "by relation (3) related to the vertex x" we often say only "by the vertex x".

In this paper, K_n, P_n, C_n denote the complete graph, path and the cycle on $\underline{n}$ vertices, respectively, and $K_{m,n}$ denotes the complete bipartite graph on $m+n$ vertices.

By the degree of a vertex $x \in (V(G)$ we mean the degree of x in the basic graph G_o.

2. SOME GENERAL RESULTS ON NORMAL DIGRAPHS

In this section we shall describe all the proper normal digraphs having as their basic graphs some particular classes of graphs (cycles, trees, unicyclic or bipartite graphs). First we have the following.

PROPOSITION 2. *Let* G *be a disconnected digraph with the components* $G_1,\dots,G_m$, *that is* $G = \bigcup_{i=1}^{m} G_i$. *Then* G *is normal if and only if all* G_i *do.*
In this case the spectrum of G *is* $\sigma(G) = \bigcup_{i=1}^{m} (G_i)$.

PROOF. If $A = A(G)$ is the adjacency matrix of G and $A_i = A(G_i)$ is the adjacency matrix of the digraph G_i $(i = 1,\dots,m)$, then we obviously have

$$A = \operatorname{diag}(A_1,\dots,A_m) \quad \text{and} \quad A' = \operatorname{diag}(A_1',\dots,A_m').$$

Hence, the condition $AA' = A'A$ is obviously equivalent to $A_iA_i' = A_i'A_i$ $(i = 1,\dots,m)$, q.e.d.

The equality $\sigma(G) = \bigcup_{i=1}^{m} \sigma(G_i)$ is then obvious.□

The previous statement shows that the consideration of normal digraphs can be reduced only to connected ones. Therefore, in the sequel we shall consider only connected digraphs.

PROPOSITION 3. *The unique proper normal digraph having the graph* C_n $(n \geq 3)$ *as its basic graph is, up to the isomorphism,, the directed cycle* $\vec{C}_n$ *(Figure 1).*

PROOF. The cycle $\vec{C}_n$ is obviously a PND. Conversely, let G be an arbitrary PND with the basic graph C_n. Starting of any its simple edge and applying Proposition 1, we easily find that G is isomorphic to the digraph $\vec{C}_n$.□

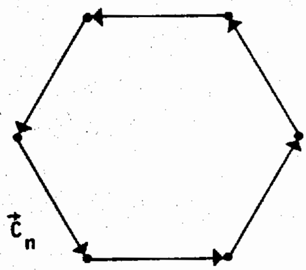

Figure 1

PROPOSITION 4. *Each normal connected tree, and each connected unicyclic digraph with at least one vertex of degree 1, is necessarily a symmetric digraph.*

PROOF. First, let G be a normal connected tree and $a \in V(G)$ be any of its vertices of degree 1. Then, obviously, the unique edge ab of G incident to a must be double, and all the edges bc of G $(c \neq a)$ must also be double. Hence, it can easily be seen that the induced subdigraph G-a of G (which is also a connected tree) is normal too, and the induction on $|G|$ completes the proof.

Since the proof of the second statement is based on the previous one, we omit it.□

As an immediate consequence of this proposition we have the following.

COROLLARY 1. *The unique normal digraph having the graph* $K_{1,n}$ $(n \geq 1)$ *as its basic graph is the symmetric digraph.*

Since the proof of the following lemma is similar to the previous ones, we also omit it.

LEMMA 1. *Let* G *be a digraph having a vertex* a *doubly adjacent to all the other vertices of* G. *Then* G *is normal if and only if the induced subdigraph* G-a *is such.*

Now we want to describe all the proper normal digraphs with the graph $K_{m,n}$ $(m,n \geq 2)$ as its basic graph. We do it only for some small values of the parameter <u>m</u>, in fact only for $m = 2,3$. The similar problem for $m \geq 4$ remains to be open. In the sequel, any white circle (and in particular, a point) denotes the induced subdigraph of a digraph G consisting only of isolated vertices. The number under a circle indicates the number of its elements. If there are all the possible edges between such two circles, and all of them are of the same kind, it is indicated by drawing only one such representing edge between these circles. First we have

PROPOSITION 5. *For any two parameters* $m,n \geq 2$ *there is at least one* PND *with the basic graph* $G_o = K_{m,n}$.

PROOF. The graph $G = T_{m,n}$ $(m,n \geq 2)$ presented in the following figure is obviously PND, and has the mentioned property, q.e.d.□

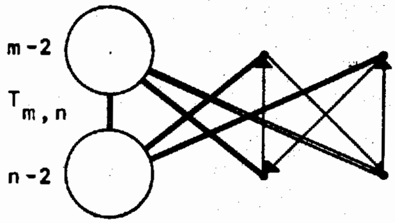

Figure 2

PROPOSITION 6. *The unique* PND *with the basic graph* $G_o = K_{2,n}$ $(n \geq 2)$ *is the digraph* $Q_{n,r}$ $(1 \leq r \leq n/2)$ *presented in the following figure:*

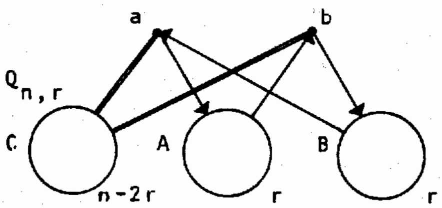

Figure 3

PROOF. The above digraph $Q_{n,r}$ is obviously PND. Conversely, assume that G is a PND with the basic graph $G_o = K_{2,n}$. Denote the 2-subset of V(G) with {a,b}. If x is any other vertex of G, then $a \to x$ implies $x \to b$, $x \to a$ implies $b \to x$, and $x \leftrightarrow a$ implies $x \leftrightarrow b$. Hence, the n-subset of V(G) can be divided into three mutually disjoint subsets

$$A = \{x \in V(G) \mid a \to x\}, \quad B = \{x \in V(G) \mid x \to a\},$$

$$C = \{x \in V(G) \mid x \leftrightarrow a\}.$$

By the vertex <u>a</u> we obiviously have that $|A| = |B| = r$, where $r \geq 1$ since G is by assumption non-symmetric. Hence, $|C| = n - 2r \geq 0$ whence $r \leq n/2$. We observe that the subset C can be empty, while the subsets A and B do not. This completes the proof.□

PROPOSITION 7. *The following two graphs* $R_{n,r}$ $(1 \leq \leq r \leq n/2)$, $S_{n,r}$ $(1 \leq r \leq n/3)$

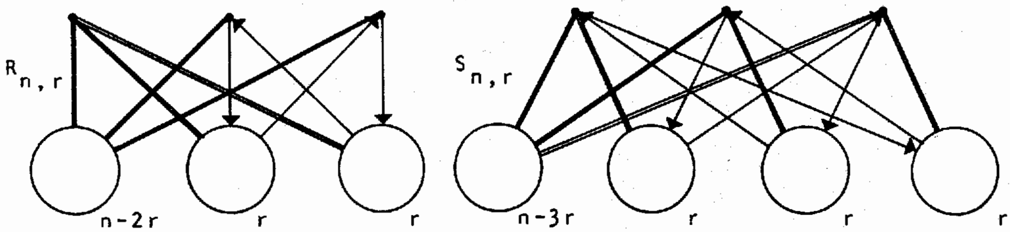

Figure 4

are the unique proper normal digraphs with the basic graph $G_o = K_{3,n}$ $(n \geq 2)$.

PROOF. The above graphs $R_{n,r}$ and $S_{n,r}$ are obviously PND. Conversely, assume that G in PND with the basic graph $G_o = K_{3,n}$ $(n \geq 2)$. Denote its 3-subset by $\{a,b,c\}$ and its n-subset by M_n. Then M_n can be divided into the following subsets:

$$A' = \{x \in M_n \mid x \leftrightarrow a,\ x \to b,\ c \to x\},$$

$$A'' = \{x \in M_n \mid x \leftrightarrow a,\ x \to c,\ b \to x\},$$

$$B' = \{x \in M_n \mid x \leftrightarrow b,\ x \to c,\ a \to x\},$$

$$B'' = \{x \in M_n \mid x \leftrightarrow b,\ x \to a,\ c \to x\},$$

$$C' = \{x \in M_n \mid x \leftrightarrow c,\ x \to a,\ b \to x\},$$

$$C'' = \{x \in M_n \mid x \leftrightarrow c,\ x \to b,\ a \to x\},$$

$$L = \{x \in M_n \mid x \leftrightarrow a,\ x \to b,\ x \to c\}.$$

By the vertices a, b and c we obviously get equations

$$(4)\qquad \begin{cases} |B'| + |C''| = |B''| + |C'|, \\ |C'| + |A''| = |C''| + |A'|, \\ |A'| + |B''| = |A''| + |B'|. \end{cases}$$

Choosing two arbitrary vertices $x \in A'$ and $y \in B''$ we have that $|A'| \cdot |B''| = 0$. Generally, we find a sequence of equalities:

(5) $$|A'|\cdot|B''| = |A''|\cdot|B'| = |A'|\cdot|C''| = |A''|\cdot|C'| = |B'|\cdot|C''| = |B''|\cdot|C'| = 0.$$

Using some simple combinatorial arguments in the system (4), (5) we conclude that only two cases can arise:

(i) $|A'| = |A''| = |B'| = |B''| = 0$, while $|C'| = |C''| = r \geq 1$;

(ii) $|A'| = |B'| = |C'| = 0$, while $|A''| = |B''| = |C''| = r \geq 1$.

In the first case we obtain that G is isomorphic to the digraph $R_{n,r}$, while in the second case G is isomorphic to the digraph $S_{n,r}$. In both cases we have that $r \geq 1$, since G is a nonsymmetric digraph.

This completes the proof.□

3. PROPER NORAML DIGRAPHS WITH AT MOST 5 VERTICES

In this section we shall describe all the proper normal digraphs having at most 5 vertices. It is done in the most economical way, with a help of computer, using the list of all connected graphs with at most 5 vertices ([2],pp.273--277), Proposition 1, and a program for the isomorphism of digraphs. Clearly, the same result could be get also by hand, but this way would spend many more time and space.

THEOREM 2. *The unique* PND *(up to the isomorphism) with at most* 5 *vertices are the digraphs* X_i (i = 1,...,14) *presented in the following figure:*

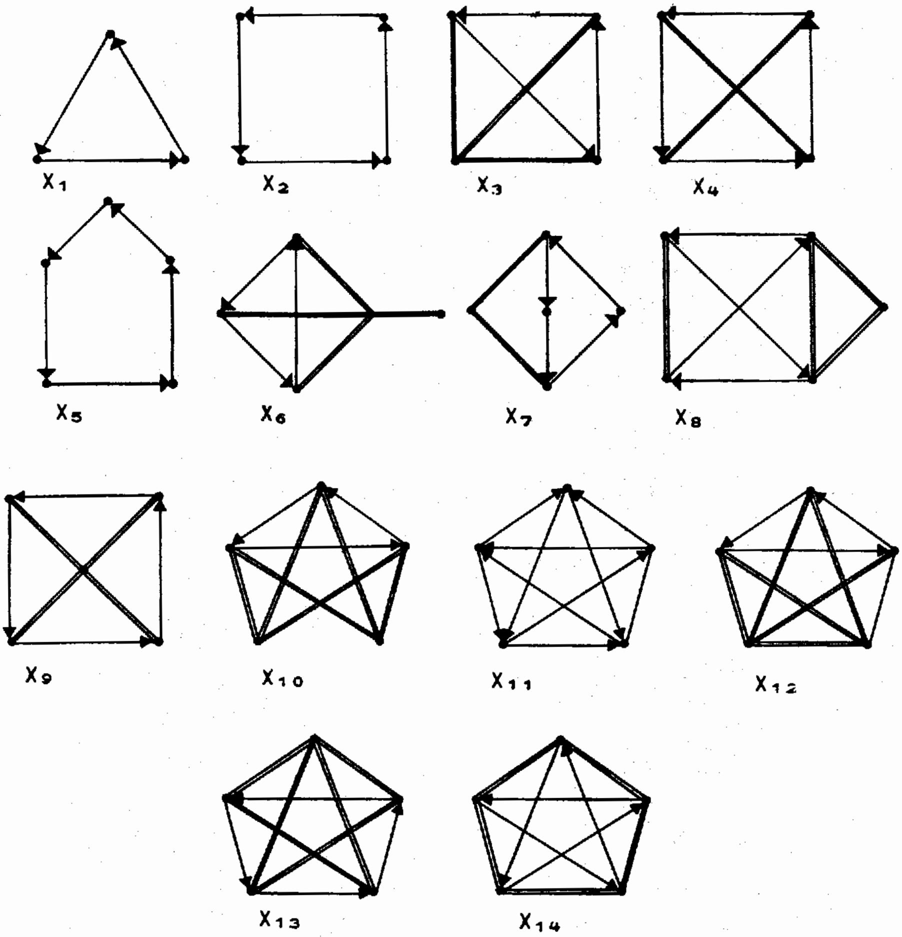

Figure 5

In the next table we give the characteristic polynominals and the spectra of all connected normal digraphs with at most 5 vertices. The number n indicates the order or a digraph.

n	digraph	CHARACTERISTIC POLYNOMINAL $P_G(\lambda)$ AND SPECTRUM
3	X_1	$\lambda^3 - 1 = (\lambda-1)(\lambda^2+\lambda+1)$ $\lambda_1 = 1,\ \lambda_{2,3} = -1/2 \pm i\sqrt{3}/2$
4	X_2	$\lambda^4 - 1 = (\lambda+1)(\lambda-1)(\lambda^2+1)$ $\lambda_1 = 1,\ \lambda_{2,3} = \pm i,\ \lambda_4 = -1$
4	X_3	$\lambda^4-3\lambda^2-4\lambda-3 = (\lambda^2+\lambda+1)(\lambda^2-\lambda-3)$ $\lambda_1 = 1/2+\sqrt{13}/2,\ \lambda_{2,3}=-1/2\pm i\sqrt{3}/2,\ \lambda_4=1/2-\sqrt{13}/2$
4	X_4	$\lambda^4-2\lambda^2-4\lambda = \lambda(\lambda-2)(\lambda^2+2\lambda+2)$ $\lambda_1 = 2,\ \lambda_2 = 0,\ \lambda_{3,4} = -1\pm i$
5	X_5	$\lambda^5-1=(\lambda-1)(\lambda^2+\frac{\sqrt{5}+1}{2}\lambda+1)(\lambda^2-\frac{\sqrt{5}-1}{2}+1)$ $\lambda_1=1,\ \lambda_{2,3}\doteq 0,3090\pm 0,9511i,\ \lambda_{4,5}\doteq -0,8090\pm 0,5878i$
5	X_6	$\lambda^5-4\lambda^3-4\lambda^2-3\lambda+1 = (\lambda^2+\lambda+1)(\lambda^3-\lambda^2-4\lambda+1)$ $\lambda_1\doteq 2,4605,\ \lambda_2\doteq 0,2391,\ \lambda_{3,4}=-1/2\pm i\sqrt{3}/2,\lambda_5\doteq -1,6996$
5	X_7	$\lambda^5-2\lambda^3-3\lambda = \lambda(\lambda^2-3)(\lambda^2+1)$ $\lambda_1 = \sqrt{3},\ \lambda_2 = 0,\ \lambda_{3,4} = \pm i,\ \lambda_5 = -\sqrt{3}$
5	X_8	$\lambda^5-4\lambda^3-6\lambda^2+4 = (\lambda^2+2\lambda+2)(\lambda^3-2\lambda^2-2\lambda+2)$ $\lambda_1\doteq 2,4812,\ \lambda_2\doteq 0,6889,\ \lambda_{3,4}=-1\pm i,\ \lambda_5\doteq -1,1701$
5	X_9	$\lambda^5-4\lambda^3-4\lambda^2-5\lambda-4 = (\lambda+1)(\lambda^2-\lambda-4)(\lambda^2+1)$ $\lambda_1=(\sqrt{17}+1)/2,\ \lambda_{2,3}=\pm i,\ \lambda_4=-1,\ \lambda_5=-(\sqrt{17}-1)/2$
5	X_{10}	$\lambda^5-6\lambda^2-7\lambda^2-6\lambda = \lambda(\lambda-3)(\lambda+2)(\lambda^2+\lambda+1)$ $\lambda_1=3,\ \lambda_2=0,\ \lambda_{3,4}=-1/2\pm i\sqrt{3}/2,\ \lambda_5=-2$
5	X_{11}	$\lambda^5-5\lambda^2-5\lambda-2 = (\lambda-2)(\lambda^4+2\lambda^3+4\lambda^2+3\lambda+1)$ $\lambda_1=2,\ \lambda_{2,3}\doteq -1/2\pm 0,3633i,\ \lambda_{4,5}\doteq -1/2\pm 1,5388i$

n	digraph	CHARACTERISTIC POLYNOMINAL $P_G(\lambda)$ AND SPECTRUM
5	X_{12}	$\lambda^5-7\lambda^3-13\lambda^2-12\lambda-5 = (\lambda+1)(\lambda^2-2\lambda-5)(\lambda^2+\lambda+1)$ $\lambda_1=1+\sqrt{6},\ \lambda_{2,3}=-1/2\pm i\sqrt{3}/2,\ \lambda_4=-1,\ \lambda_5=1-\sqrt{6}$
5	X_{13}	$\lambda^5-6\lambda^3-12\lambda^2-8\lambda = \lambda(\lambda^2-2\lambda-4)(\lambda^2+2\lambda+2)$ $\lambda_1=1+\sqrt{5},\ \lambda_2=0,\ \lambda_{3,4}=-1\pm i,\ \lambda_5=1-\sqrt{5}$
5	X_{14}	$\lambda^5-5\lambda^3-10\lambda^2-5\lambda-3 = (\lambda-3)(\lambda^4+3\lambda^3+4\lambda^2+2\lambda+1)$ $\lambda_1=3,\ \lambda_{2,3}\doteq-0{,}1910\pm0{,}5878i,\ \lambda_{4,5}\doteq-1{,}3090\pm0{,}9511i$

REMARK. In a subsequent paper the author describes all proper normal digraphs whose degrees of all vertices do not exceed 3. A student of the author also describes all proper normal digraphs with exactly 6 vertices.

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REZIME

SPEKTAR JEDNOG NORMALNOG DIGRAFA

U ovom radu posmatramo spektre konačnih digrafa čija je 0-1 matrica susedstva normalna ali ne i simetrična. Dokazu-

jemo izvesne opšte osobine ovakvih digrafova, nalazimo sve normalne digrafove sa najviše 5 čvorova i izračunavamo njihove spektre.

Received by the editors February 3, 1987.