ZBORNIK RADOVA
Prirodno-matematičkog fakulteta
Univerziteta u Novom Sadu
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A GENERALIZATION OF A SECTION THEOREM OF KY FAN AND ITS APPLICATIONS TO VARIATIONAL INEQUALITIES

A. Chitra and P.V. Subrahmanyam

Department of Mathematics, Indian Institute of Technology, 600036 Madras, India

ABSTRACT

In this paper some section theorems are obtained. These theorems are generalizations of section theorems from [8] and [13]. Some applications are also given.

1. INTRODUCTION

In [8] Ky Fan generalized the KKM mapping theorem to infinite dimensional spaces and obtained as a consequence a section theorem leading to a proof of Tychonoff's fixed point theorem. Itoh, Takahashi and Yanagi [13] gave an elementary proof of this section theorem based on Brouwer's fixed point theorem and also obtained an analogue of this section theorem wherefrom they established a few existence theorems for complementarity problems as well as nonlinear variational inequalities involving uppersemicontinuous multifunctions. In [21] Takahashi obtained a basic lemma leading to an extension

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of a variational inequality studied by Browder [4]. In § 3 we have obtained a generalization of Ky Fan's section theorem and it leads to the formulation of a class of variational inequalities subsuming the one considered by Takahashi [21]. An extension of a coincidence theorem of Jiang Jiahe [14] is yet another application of our theorem. In § 4 another section theorem generalizing that of Itoh, Takahashi and Yanagi [13] is proved along with the generalization of some results on variational inequalities obtained by these authors.

For applications of fixed point theorems in the solutions of complementarity problems, variational inequalities and quasivariational inequalities Allen [1], Browder [3], Baiocchi and Capelo [2], Coppoletta [5], Dugundji and Granas [6], Gwinner [11], Karamardian [16], Juberg and Karamardian [15], Kinderlehrer and Stampacchia [17], Minty [18], More [19] and Mosco [20] may be referred.

2. PRELIMINARIES

Throughout the paper E denotes a Hausdorff topological vector space, 2^E the collection of nonempty subsets of E. For $H, K \subseteq E$ the boundary and interior of K relative to H are defined as $B_H(K) = \overline{K} \cap (\overline{H-K})$ and $I_H(K) = K \cap (B_H(K))^c$ respectively where $\overline{A}$ denotes the closure of A and A^c the complement of A. CK(E) denotes the set of all nonempty compact, convex subsets of E. For any pair of topological vector spaces E and F denote by $\langle .,. \rangle$ a bilinear form of $F \times E$ into the reals $\mathbb{R}$ and by $\mathbb{R}^-$ the set of nonpositive real numbers. For any locally convex space E we assume that the topology of E^* (the dual of E) is the strong topology and $\langle w,x \rangle$ is the value of $w \in E^*$ at $x \in E$. For any cone H in a topological vector space E, i.e., H is a closed convex subset of E such that $\alpha x + \beta y \in H$ for all $\alpha, \beta \geq 0$ and $x, y \in H$, the polar H^* of H is the cone defined by

$$H^* = \{ y \in E^* : \langle y,x \rangle \geq 0 \text{ for all } x \in H \}.$$

Ky Fan [10] introduced the following concept of an upperdemicontinuous maps that include the class of uppersemicontinuous maps in real spaces.

Definition 2.1. *Let* E *be a real locally convex space and* T *a multimap of* $S \subset E$ *into* 2^E *is said to be upperdemicontinuous if for every* $x \in S$ *and every half space* H *containing* Tx *there exists a neighbourhood of* x *whose image under* T *is contained in* H *where* H *is of the form* $\{x \in E : h(x) > r\}$ *where* h *is a continuous linear functional not identically zero and* r *any real number.*

The basic section theorem of Ky Fan whose applications can be found in Ky Fan [9], and Iohvidov [12] is as follows.

Theorem 2.1. (Ky Fan [8]) *Let* K *be a nonempty compact convex subset of a Hausdorff topological vector space* E. *Let* A *be a closed subset of* $K \times K$ *for which the following conditions hold:*

(i) $(x,x) \in A$ *for every* $x \in K$,

(ii) *For each* $x \in K$, $\{y \in K : (x,y) \notin A\}$

is convex or empty.

Then there exists $x_0 \in K$ *such that* $\{x_0\} \times K \subseteq A$.

The following results on variational inequalities and complementarity problems have been obtained in Takahashi [21].

Theorem 2.2. (Takahashi [21]) *Let* H *be a closed convex subset of a locally convex Hausdorff topological vector space* E *and* T *a continuous mapping of* H *into* E^*. *If there exists a compact convex set* K *of* H *such that* $I_H(K) \neq \phi$ *and for each* $z \in B_H(K)$ *there is* $u_0 \in I_H(K)$ *with* $\langle Tz, z-u_0 \rangle \geq 0$, *then there exists* $x_0 \in H$ *such that* $\langle Tx_0, x-x_0 \rangle \geq 0$ *for all* $x \in H$.

Theorem 2.3. (Takahashi [21]) *Let* H *be a cone in a locally convex space* E *and* T *a continuous mapping of* H *into* E^*.

If there exists a compact convex set K *of* H *such that* $I_H(K) \neq \phi$ *and for each* $z \in B_H(K)$ *there is* $u_0 \in I_H(K)$ *with* $\langle Tz, z-u_0 \rangle \geq 0$ *then there exists* $x^* \in H$ *such that* $Tx^* \in H^*$ *and* $\langle Tx^*, x^* \rangle = 0$.

A coincidence theorem of Jiang Jiahe [14] for upper-demicontinuous multimaps is given below.

Theorem 2.4. (Jiang Jiahe [14]) *Let* K *be a compact convex set in a locally convex space* E. *Let* $F, G : K \to 2^E$ *be two upper-demicontinuous mappings such that for any* $x \in K$, F(x) *and* G(x) *are nonempty closed convex sets and either* F(x) *or* G(x) *is compact. Suppose that for any* $x \in E$ *and any continuous linear functional* ϕ *on* E $\inf_{y \in E} \varphi(x-y) \geq 0$ *implies* $\inf_{\substack{u \in F(x) \\ v \in G(x)}} \varphi(u-v) \geq 0$. *Then there exists* $x_0 \in K$ *such that* $F(x_0) \cap G(x_0) \neq \emptyset$.

The following is a section theorem analogous to Theorem 2.1 where we have distinct topological vector spaces.

Theorem 2.5. (Itoh, Takahashi and Yanagi [13]) *Let* K_1 *be a nonempty compact convex subset of a locally convex space* E *and* K_2 *a closed convex subset of a Hausdorff topological vector space* F. *Let* A *be a subset of* $K_1 \times K_2$ *having the following properties:*

(i) *A is closed*

(ii) *For any* $y \in K_2$, $\{x \in K_1 : (x,y) \in A\}$ *is nonempty and convex*

(iii) *For any* $x \in K_1$, $\{y \in K_2 : (x,y) \notin A\}$ *is convex or empty.*

Then there exists $x_0 \in K_1$ *such that* $\{x_0\} \times K_2 \subseteq A$.

Some results on variational inequalities and complementarity problems proved in [13] have been stated below.

Theorem 2.6. (Itoh, Takahashi and Yanagi [13]) *Let* H *be a nonempty closed convex subset of a Haudorff topological vector space* E, F *locally convex. Let* $T : H \to CK(F)$ *be an upper-*

semicontinuous map and $f : H \to \mathbb{R}$ *a lower semicontinuous convex function. Suppose that there exists a nonempty compact convex set* K *of* H *with* $I_H(K) \neq \phi$ *such that* $<.,.>$ *is jointly continuous on* $F \times K$ *and for each* $z \in B_H(K)$ *there is* $u \in I_H(K)$ *for which* $\inf_{w \in T(z)} < w, z-u > \geq f(u) - f(z)$.

Then there exists $x_0 \in K$ *and* $w_0 \in T(x_0)$ *such that* $< w_0, x-x_0 > \geq f(x_0) - f(x)$ *for every* $x \in H$.

Theorem 2.7. (Itoh, Takahashi and Yanagi [13]) *Let* H *be a cone in the real n dimensional space* $\mathbb{R}^n$ *and* $T : H \to \to CK(\mathbb{R}^n)$ *an uppersemicontinuous mapping for which there is a constant* $c > 0$ *such that* $< w-v, x > \geq c\|x\|^2$ *for all* $x \in H$, $w \in T(x)$ *and* $v \in T(0)$. *Then there exists* $x_0 \in H$ *and* $w_0 \in T(x_0)$ *such that* $w_0 \in H^*$ *and* $< w_0, x_0 > = 0$.

3. A GENERALIZATION OF A SECTION THEOREM OF KY FAN AND ITS APPLICATIONS

The following theorem includes Theorem 2.1.

Theorem 3.1. *Let* K *be a nonempty compact, convex subset of a Hausdorff topological vector space* E. *Let* $A \subseteq K \times K$ *and* $g : K \to K$ *such that the following conditions are satisfied.*

(i) *For every* $y \in K$, $\{x \in K : (gx,y) \in A\}$ *is closed*

(ii) *For every* $x \in K$, $(gx,x) \in A$,

(iii) *For each* $x \in K$, $\{y \in K : (gx,y) \notin A\}$ *is empty or convex.*

Then there exists $x_0 \in K$ *such that* $\{gx_0\} \times K \subseteq A$.

Proof. Suppose that for any $x \in K$, there exists $y \in K$ such that $(gx,y) \notin A$. For each $y \in K$, let $A(y) = \{x \in K : (gx,y) \notin A\}$ then we have $K = \bigcup_{y \in K} A(y)$. By (i)

$A(y)$ is open in K for each $y \in K$. Since K is compact there exists a finite number of points $\{y_1, y_2, \ldots, y_n\}$ of K such that

$$K = \bigcup_{i=1}^{n} A(y_i).$$

Let $\{\beta_1, \beta_2, \ldots, \beta_n\}$ be a partition of unity corresponding to this covering i.e., each β_i is a continuous mapping of K into $[0,1]$ which vanishes outside of $A(y_i)$ while

$$\sum_{i=1}^{n} \beta_i(x) = 1$$

for every $x \in K$. We define a mapping $p : K \to K$ by

$$p(x) = \sum_{i=1}^{n} \beta_i(x) y_i.$$

Then p maps the simplex S spanned by the set $\{y_1, y_2, \ldots, y_n\}$ into itself, p has a fixed point $z \in S$ by Brouwer's fixed point theorem. For every i with $\beta_i(z) > 0$, $(g(z), y_i) \notin A$. Thus by (iii) we get

$$(g(z), p(z)) = (g(z), \sum_{i=1}^{n} \beta_i(z) y_i) \notin A.$$

On the other hand $(g(z), p(z)) = (g(z), z) \in A$ by (ii). This is a contradiction. Therefore there exists $x_0 \in K$ such that $\{gx_0\} \times K \subseteq A$.

Remark 3.1. When g = identity map on K we get Theorem 2.1.

As an example which illustrates Theorem 3.1 we have the following.

Example 3.1. Let $K = [0,1] \subseteq \mathbb{R}$ and $g : K \to K$ be defined as $gx = 1-x$ and $A = \{(x,y) \in K \times K : y \leq 1-x\}$. Then it can be easily verified that g and A satisfy all the conditions of Theorem 3.1. Clearly $x_0 = 1$ is such that $\{gx_0\} \times K \subseteq A$. We do not require the diagonal $\{(x,x) : x \in K\}$ to be contained in A.

As an application of Theorem 3.1 we have the following in which for g = identity map on K, the result is due to Takahashi ([21], Lemma 1).

Theorem 3.2. *Let E be a Hausdorff topological vector space and $K \subseteq E$ compact and convex $g : K \to K$ and $F : K \times K \to \mathbb{R}$ are maps with the following conditions:*

(i) *For every $x \in K$, $y \mapsto F(gx,y)$ is convex,*

(ii) $F(gx,x) \geq c$ *for every* $x \in K$ *for some real* c,

(iii) *For every $y \in K$, $x \mapsto F(gx,y)$ is uppersemicontinuous.*

Then there exists $x_0 \in K$ such that $F(gx_0,x) \geq c$ for every $x \in K$.

Proof. Let $A = \{(x,y) \in K \times K : F(x,y) \geq c\}$. It can be easily verified that under assumptions (i), (ii) and (iii) all the conditions of Theorem 3.1.are satisfied. Hence there exists $x_0 \in K$ such that $\{gx_0\} \times K \subseteq A$ i.e., $F(gx_0,x) \geq c$ for every $x \in K$.

With the help of Theorem 3.2 we prove the following result.

Theorem 3.3. *Let E be a locally convex Hausdorff topological vector space, $T : K \to E^*$ and $g : K \to K$ are maps where $K \subseteq E$ is compact and convex with the following conditions.*

(i) $x \mapsto \langle Tgx, y-gx \rangle$ *is uppersemicontinuous for every* $y \in K$,

(ii) $x \mapsto \langle Tgx, x-g(x) \rangle$ *is lower semicontinuous.*

Then there exist $x_0, y_0 \in K$ such that for every $y \in K$

$\langle Tg(x_0), y-g(x_0) \rangle \geq \langle Tg(y_0), y_0-g(y_0) \rangle$.

Proof. Define $F : K \times K \to \mathbb{R}$ by $F(x,y) = \langle Tx, y-x \rangle$. Under the assumptions (i) and (ii) F satisfies the conditions of Theorem 3.2 with $c = \inf_{x\in K} \langle Tg(x), x-g(x) \rangle$ the infimum being attained at some $y_0 \in K$ in view of the compactness of K and (ii). So by Theorem 3.2 there exists $x_0 \in K$ such that $F(gx_0, y) \geq \langle Tg(y_0), y_0 - g(y_0) \rangle$ for every $y \in K$, i.e., $\langle Tg(x_0), y-g(x_0) \rangle \geq \langle Tg(y_0), y_0 - g(y_0) \rangle$ for every $y \in K$.

Remark 3.2. When g reduces to the identity map on K and $T : K \to E^*$ continuous we get a result of Browder [4].

We extend below Theorem 3.3 to closed convex sets.

Theorem 3.4. *Let* H *be a closed convex subset of a locally convex Hausdorff topological vector space* E, $T : H \to E^*$ *and* $g : K \to K$ *where* $K \subseteq H$ *is compact and convex with* $I_H(K) \neq \phi$. *Suppose (i)* $x \mapsto \langle Tg(x), y-g(x) \rangle$ *is uppersemicontinuous for every* $y \in K$, *(ii)* $x \mapsto \langle Tg(x), x-g(x) \rangle$ *is lower semicontinuous and (iii) for every* $g(z) \in B_H(K)$ *there exists* $u_0 \in I_H(K)$ *with* $\langle T(gz), u_0 - g(z) \rangle \leq \langle Tg(x), x-g(x) \rangle$ *for every* $x \in K$. *Then there exist* $x_0, y_0 \in K$ *such that* $\langle T(g(x_0)), y-g(x_0) \rangle \geq \langle T(g(y_0)), y_0 - g(y_0) \rangle$ *for every* $y \in H$.

Proof. By Theorem 3.3 there exist $x_0, y_0 \in K$ such that $\langle Tg(x_0), y-g(x_0) \rangle \geq \langle Tg(y_0), y_0 - g(y_0) \rangle$ for every $y \in K$ where $\langle Tg(y_0), y_0 - g(y_0) \rangle = \inf_{x\in K} \langle Tg(x), x-g(x) \rangle$. If $gx_0 \in I_H(K)$ for every $y \in H$, there exists $\lambda \in (0,1)$ with $\lambda y + (1-\lambda) gx_0 \in K$. Thus $\langle Tg(x_0), \lambda y + (1-\lambda) gx_0 - gx_0 \rangle \geq \langle Tg(y_0), y_0 - g(y_0) \rangle$, i.e., $\langle Tg(x_0), y-g(x_0) \rangle \geq \langle Tg(y_0), y_0 - g(y_0) \rangle$. If $gx_0 \in B_H(K)$ there exists $u_0 \in I_H(K)$ with

$$\text{(I)} \qquad \inf_{x\in K} \langle Tg(x), x-g(x) \rangle \geq \langle Tg(x_0), u_0 - g(x_0) \rangle$$

by hypothesis. For every $y \in H$ there exists $\lambda \in (0,1)$ with $\lambda y + (1-\lambda) u_0 \in K$. Hence $\langle Tg(x_0), \lambda y + (1-\lambda) u_0 - g(x_0) \rangle \geq$ $\geq \langle Tg(y_0), y_0 - g(y_0) \rangle$ i.e., $\lambda \langle Tg(x_0), y-gx_0 \rangle \geq \langle Tg(y_0), y_0 - g(y_0) \rangle + (1-\lambda)(\langle Tg(x_0), g(x_0) - u_0 \rangle)$. By (I) we conclude

$< Tg(x_0), y-g(x_0) > \geq < Tg(y_0), y_0-g(y_0) >$ for every $y \in H$.

When g = identity map on K, and $T : H \to E^*$ is continuous we get Theorem 2.2.

The following nonlinear complementarity problem is a consequence of Theorem 3.4.

Theorem 3.5. *Let* $H \subseteq E$ *be a cone where* E *is a locally convex Hausdorff topological vector space. If* $K \subseteq H$ *is compact and convex,* $T : H \to E^*$, $g : K \to K$ *are maps with the conditions:*

(i) $x \mapsto < Tg(x), y-g(x) >$ *is uppersemicontinuous for every* $y \in K$,

(ii) $x \mapsto < Tg(x), x-g(x) >$ *is lowersemicontinuous,*

(iii) *For every* $g(z) \in B_H(K)$ *there exists* $u_0 \in I_H(K)$ *with* $< Tg(z), u_0-g(z) > \leq < Tg(x), x-g(x) >$ *for every* $x \in K$.

Then there exist $x_0, y_0 \in K$ *with* $|< Tg(x_0), g(x_0) >| \leq$ $\leq < Tg(y_0), y_0-g(y_0) >$ *and* $Tg(x_0) \in H^*$.

Proof. By Theorem 3.4 there exist $x_0, y_0 \in K$ with $< Tg(x_0), y-gx_0 > \geq < Tg(y_0), y_0-g(y_0) >$ for every $y \in H$. Taking $y = 0$ and $2\, gx_0$ we get $|< Tg(x_0), g(x_0) >| \leq < Tg(y_0), y_0-g(y_0) >$ and $< Tg(x_0), y > \geq 0$ for every $y \in H$ implying that $Tg(x_0) \in H^*$.

Remark 3.3 When g is the identity map on K and $T : H \to E^*$ is continuous we get Theorem 2.3.

The following is an improvement of a result of Jiang Jiahe [14].

Theorem 3.6. *Let* K *be a compact convex subset of a locally convex space* E, F *and* $G : K \to 2^K$ *are upperdemicontinuous multimaps such that* $F(x)$ *and* $G(x)$ *are nonempty closed convex subsets of* E *for each* $x \in K$ *with the condition that either* $F(gx)$ *or* $G(x)$ *is compact for each* $x \in K$ *where* $g : K \to K$ *is a continuous map. If for every continuous linear functional* φ

and $0 < t < 1$ $\varphi(gy_i - gx) > 0$ $(i = 1,2)$ *implies that* $\varphi(g(ty_1 + (1-t)y_2) - g(x)) > 0$ (I) *and for every* $x \in g(K)$ *with* $\inf_{y \in g(K)} \varphi(x-y) \geq 0$ *implies that* $\inf_{\substack{u \in F(gx) \\ v \in G(x)}} \varphi(u-v) \leq 0$. *Then there exists* $x_0 \in K$ *such that* $F(gx_0) \cap G(x_0) \neq \phi$.

Proof. Suppose that the conclusion does not hold. Then for any $x \in K$, $F(gx)$ and $G(x)$ can be strictly separated by a closed hyperplane by separation theorem in a locally convex space i.e., there are real numbers r_x and a nonzero continuous linear functional φ_x on E such that $\varphi_x(u) > r_x > \varphi_x(v)$ for any $u \in F(gx)$ and $v \in G(x)$. By the continuity of g and upperdemicontinuity of F and G there is a neighbourhood U_x of x in K such that $U_x \subseteq \{y \in K :$ for every $u \in F(gy)$ and $v \in G(y), \varphi_x(u) > r_x > \varphi_x(v)\}$. Since at least one of $F(Gy)$ or $G(y)$ is compact say $F(g(y))$ is compact, there is a real number s_x such that for any $u \in F(gy)$ and $v \in G(y), \varphi_x(u) \geq s_x > r_x > \varphi_x(v)$. Hence $\varphi_x(u-v) \geq s_x - r_x > 0$. Consequently

$$U_x \subseteq \left\{y \in K : \inf_{\substack{u \in F(gy) \\ v \in G(y)}} \varphi_x(u-v) > 0\right\}.$$

By the compactness of K there exists $\{x_1, x_2, \ldots, x_n\} \subseteq K$ such that $\bigcup_{i=1}^{n} U_{x_i} = K$. Let $\{\beta_i\}_{i=1}^{n}$ be the corresponding partition of unity. Define A as follows. $A = \{(x,y) \in K \times K : \sum_{i=1}^{n} \beta_i(x)\varphi_{x_i}(gy - x) \leq 0\}$. Since φ_{x_i}, β_i and g are continuous for every $y \in K$, the set $\{x \in K : (gx,y) \in A\}$ is closed and $(gx,x) \in A$ for every $x \in K$. For each $x \in K$, $\{y \in K : (gx,y) \notin A\}$ is convex in view of (I) for the continuous linear functional $\varphi = \sum_{i=1}^{n} \beta_i(gx)\varphi_{x_i}$. Thus all the conditions of Theorem 3.1 are satisfied. Hence there exists $x_0 \in K$ such that $\{gx_0\} \times K \subseteq A$ i.e., $\sum_{i=1}^{n} \beta_i(gx_0)\varphi_{x_i}(gy - gx_0) \leq 0$ for every $y \in K$.

For $\varphi = \sum_{i=1}^{n} \beta_i(gx_0)\varphi_{x_i}$ a continuous linear functional we have $\varphi(gx_0 - gy) \geq 0$ for every $y \in K$ and hence $\inf_{z \in g(K)} \varphi(gx_0 - z) \geq 0$.

But for every i with $\beta_i(gx_o) > 0$, $gx_o \in U_{x_i}$ where

$$\inf_{\substack{u\in F(g(z))\\ v\in G(z)}} \varphi_{x_i}(u-v) > 0$$

where $z = gx_o$. Thus

$$\inf_{\substack{u\in F(g(z))\\ v\in G(z)}} \varphi(u-v) \geq \sum_{i=1}^{n} \beta_i(gx_o) \inf_{\substack{u\in F(g(z))\\ v\in G(z)}} \varphi_{x_i}(u-v) > 0$$

which contradicts our hypothesis. Hence there exists $x_o \in K$ with $F(gx_o) \cap G(x_o) \neq \phi$.

Remark 3.4. When g reduces to the identity map on K we get Theorem 2.4 due to Jiang Jiahe [14].

4. GENERALIZATION OF A SECTION THEOREM DUE TO ITOH, TAKAHASHI AND YANAGI AND A CLASS OF VARIATIONAL TYPE INEQUALITIES

When we have distinct topological vector spaces the following is a generalization of Theorem 2.5.

Theorem 4.1. *Let* K_1 *be a nonempty compact, convex subset of a locally convex space* E *and* K_2 *a nonempty closed convex subset of a Hausdorff topological vector space* F. *Let* A *be a subset of* $K_1 \times K_2$ *and* g *a continuous self map on* K_1 *such that the following conditions are satisfied:*

(i) *A is closed*

(ii) *For any* $y \in K_2$, $\{x \in K_1 : (gx,y) \in A\} \neq \phi$ *and convex*

(iii) *For any* $x \in K_1$, $\{y \in K_2 : (gx,y) \notin A\}$ *is empty or convex*

Then there exists $x_o \in K$ *with* $\{gx_o\} \times K_2 \subseteq A$.

Proof. Suppose that the assertion of the theorem is false. Then, for each $x \in K_1$, there is $y \in K_2$ such that

$(gx,y) \notin A$. Denote $A(y) = \{x \in K_1 : (gx,y) \notin A\}$ for any $y \in K_2$, then there exists a finite covering $\{A(y_i)\}_{i=1}^n$ of K and a partition of unity $\{\beta_i\}_{i=1}^n$ corresponding to this covering. Set $p(x) = \sum_{i=1}^{n} \beta_i(x)y_i$ for any $x \in K_1$. Then p is a continuous mapping of K_1 into K_2. Define a mapping $T : K_1 \to 2^{K_1}$ by $T(x) = \{u \in K_1 : (g(u),p(x)) \in A\}$ then by (i) and (ii) $T(x)$ is non-empty convex and compact for every $x \in K_1$ (By the continuity of g and closedness of A, $T(x)$ is a closed subset of K_1 for every $x \in K_1$). Using the continuity of g, p and (i) it can be verified that T is uppersemicontinuous. Hence T has a fixed point $z \in K_1$ by Fan's fixed point theorem [7]. Thus $(g(z),p(z)) \in A$. On the other hand by (iii) $(g(z),p(z)) \notin A$. This contradiction proves the theorem.

When g = identity mapping on K_1 we obtain Theorem 2.5.

Throughout the rest of this section H is a closed convex subset of a Hausdorff topological vector space E and F a Hausdorff locally convex space, $K \subseteq H$ a compact convex set with $I_H(K) \neq \phi$. g maps H continuously into itself leaving K invariant, $f : H \to \mathbb{R}$ a real valued continuous convex function and $T : H \to CK(F)$. The bilinear form $\langle\cdot,\cdot\rangle$ is jointly continuous on $F \times K$.

Theorem 4.2. *Suppose the following conditions are satisfied for the maps* ψ_i $(i = 1,2,3)$ *of* H *into itself leaving* K *invariant.*

(i) *The maps* $\psi_2 g$ *and* ψ_1 *are continuous where* ψ_1 *is affine and* Tg *is uppersemicontinuous on* K.

(ii) *For every* $x \in K$, *there exists* $w \in T(g(x))$ *such that* $\langle w,\psi_1(x) - \psi_2(gx)\rangle \geq f(gx) - f(x)$.

(iii) *For* $\psi_3(x) \in B_H(K)$ *there exists* $y \in I_H(K)$ *such that*
$$\inf_{w\in T(gx)} \langle w,\psi_2(gx) - \psi_1(y)\rangle \geq f(y) - f(gx).$$

(iv) $fg \geq f\psi_3$ *and* $\psi_2 \cdot g = \psi_1 \cdot \psi_3$.

Then there exists $x_0 \in K$ *and* $w_0 \in T(gx_0)$ *such that* $< w_0, \psi_1(x) - \psi_2(gx_0) > \geq f(gx_0) - f(x)$ *for every* $x \in H$.

Proof. Let $A = \{(x,y) \in K \times K : \sup_{w \in T(x)} < w, \psi_1(y) - \psi_2(x) > \geq f(x) - f(y)$. By (ii) $(gx,x) \in A$ for every $x \in K$ and thus A is nonempty. Since ψ_1 is affine and f is convex, $\{y \in K : (gx,y) \notin A\}$ is convex for every $x \in K$ and by the continuity of $\psi_2 g$ and the uppersemicontinuity of Tg, $\{x \in K : (gx,y) \in A\}$ is closed for every $y \in K$. Thus all the conditions of Theorem 3.1 are satisfied. Hence there exists $x_0 \in K$ such that $\{gx_0\} \times K \subseteq A$. i.e.,

$$\text{(I)} \qquad \sup_{w \in T(gx_0)} < w, \psi_1(x) - \psi_2(gx_0) > \geq f(gx_0) - f(x)$$

for every $x \in K$.

Now, define $B = \{(w,x) \in T(gx_0) \times K : < w, \psi_1(x) - \psi_2(gx_0) > \geq f(gx_0) - f(x)\}$. B is nonempty in view of (I). Since ψ_1 and f are continuous, B is closed. By (I) and the affineness of ψ_1 it can be verified that B satisfies all the conditions of Theorem 4.1. (with g = identity in that theorem). So there exists $w_0 \in T(gx_0)$ such that $\{w_0\} \times K \subseteq B$. i.e.,

$$\text{(II)} \qquad < w_0, \psi_1(x) - \psi_2(gx_0) > \geq f(gx_0) - f(x)$$

for every $x \in K$.

If $\psi_3(x_0) \in I_H(K)$, for each $x \in H$ we can choose λ $(0 < \lambda < 1)$ so that $\lambda x + (1 - \lambda)\psi_3(x_0) \in K$. By II

$$< w_0, \psi_1(x) - \psi_2(gx_0) > \geq \lambda(f(gx_0) - f(x)) +$$
$$+ (1 - \lambda)[f(gx_0) - f(\psi_3(x_0)) + < w_0, \psi_2(gx_0) -$$
$$- \psi_1\psi_3(x_0) >]$$

By (iv) $< w_0, \psi_1(x) - \psi_2(gx_0) > \geq f(gx_0) - f(x)$ for every $x \in H$.

Suppose $\psi_3(x_0) \in B_H(K)$, by (iii) there exists $y_0 \in I_H(K)$ such that $\inf_{w \in T(gx_0)} < w, \psi_2(gx_0) - \psi_1(y_0) > \geq f(y_0) - f(gx_0)$.

Since $w_0 \in T(gx_0)$, $< w_0, \psi_2(gx_0) - \psi_1(y_0) > \geq f(y_0) - f(gx_0)$. Now if $x \in H$, for $y_0 \in I_H(K)$ there exists $\lambda \in (0,1)$ such that $\lambda x + (1-\lambda)y_0 \in K$. By (II) $< w_0, \psi_1(\lambda x + (1-\lambda)y_0) - \psi_2(gx_0) > \geq f(gx_0) - f(\lambda x + (1-\lambda)y_0)$ from which it follows that

$$\lambda < w_0, \psi_1(x) - \psi_2(gx_0) > \geq \lambda(f(gx_0) - f(x))$$

$$+ (1-\lambda)[f(gx_0) - f(y_0) + < w_0, \psi_2(gx_0) - \psi_1(y_0) >]$$

By (iii) and (iv) $< w_0, \psi_1(x) - \psi_2(gx_0) > \geq f(gx_0) - f(x)$ for every $x \in H$.

Remark 4.1. When T is single valued the lowersemicontinuity of f is sufficient to draw the conclusion of Theorem 4.2.

For particular choices of ψ_1, ψ_2 and ψ_3 we obtain the following results.

Theorem 4.3. *Suppose the following conditions hold:*

(i) T *is uppersemicontinuous*

(ii) *For* $x \in B_H(K)$, *there exists* $y \in I_H(K)$ *such that* $\inf_{w \in T(x)} < w, x-y > \geq f(y) - f(x)$.

Then there exists $x_0 \in K$ *and* $w_0 \in T(x_0)$ *such that* $< w_0, x - x_0 > \geq f(x_0) - f(x)$ *for every* $x \in H$.

Proof. In Theorem 4.2 set $\psi_1 = \psi_2 = \psi_3 = g =$ identity on H. It can be verified that all conditions of Theorem 4.2 are satsified under the assumptions (i) and (ii) and we have the required result.

Remark 4.2. In the proof of Theorem 2.6, though the authors have assumed f to be merely lowersemicontinuous, it appears that not only the lowersemicontinuity of f, but also that of - f (which leads to the continuity of f) is indispensable. Theorem 4.3 is a modification of Theorem 2.6.

Theorem 4.4. *Suppose we have the following conditions.*

(i) Tg *is uppersemicontinuous on* K,

(ii) *For every* $x \in K$, *there exists* $w \in T(gx)$ *such that* $< w, x-g^2x > \geq f(gx) - f(x)$,

(iii) *For* $g^2(x) \in B_H(K)$ *there exists* $y \in I_H(K)$ *such that* $\inf\limits_{w \in T(gx)} < w, g^2x-y > \geq fy - f(gx)$ *and*

(iv) $fg \geq fg^2$.

Then there exists $x_0 \in K$ *and* $w_0 \in T(gx_0)$ *such that* $<w_0, x-g^2(x_0) > \geq f(gx_0) - f(x)$ *for every* $x \in H$.

Proof. In Theorem 4.2, put ψ_1 = identity on H, $\psi_2 = g$ and $\psi_3 = g^2$. For this choise of ψ_i $(i = 1,2,3)$ under the assumptions (i) - (iv) it can be verified that all the conditions of Theorem 4.2 hold. Hence there exists $x_0 \in K$ and $w_0 \in T(gx_0)$ such that $< w_0, x-g^2x_0 > \geq f(gx_0) - f(x)$ for every $x \in H$.

Theorem 4.5. *Suppose we have the following assumptions on* T *and* g.

(i) g *is affine and* Tg *is uppersemicontinuous on* K

(ii) *For every* $x \in K$, *there exists* $w \in T(gx)$ *such that* $< w, gx-g^2x > \geq f(gx) - f(x)$,

(iii) *For* $gx \in B_H(K)$ *there exists* $y \in I_H(K)$ *such that* $\inf\limits_{w \in T(gx)} < w, g^2x - gy > \geq f(y) - f(gx)$.

Then there exists $x_0 \in K$ *and* $w_0 \in T(gx_0)$ *such that* $< w_0, gx - g^2x_0 > \geq f(gx_0) - f(x)$ *for every* $x \in H$.

Proof. In Theorem 4.2 take $\psi_1 = \psi_2 = \psi_3 = g$. Under the given hypotheses, for the particular choice of ψ_i, it can be verified that all the conditions of Theorem 4.2 are satisfied. So there exists $x_0 \in K$ and $w_0 \in T(gx_0)$ such that $< w_0, gx - g^2x_0 > \geq f(gx_0) - f(x)$ for every $x \in H$.

For convenience we shall assume in the rest of the section that Tg is uppersemicontinuous on H.

Theorem 4.2 still holds if in the place of F we take E^*, the dual of E a locally convex space and denote by $< w,x >$ the value of $w \in E^*$ at $x \in E$.

Theorem 4.6. *If* H *is a closed convex subset of a locally convex space* $E, K \subseteq H$ *compact and convex with* $I_H(K) \neq \phi$. *If* g, ψ_i $(i = 1,2,3)$ *are maps of* H *into itself leaving* K *invariant where* g *is continuous,* $f : H \to \mathbb{R}$ *a continuous convex function,* $T : H \to CK(E^*)$ *with*

(i) *The maps* $\psi_2 g$ *and* ψ_1 *are continuous where* ψ_1 *is affine and* Tg *is uppersemicontinuous,*

(ii) *For every* $x \in K$*, there exists* $w \in T(gx)$ *such that* $< w,\psi_1(x) - \psi_2(gx) > \geq f(gx) - f(x)$,

(iii) *For* $\psi_3(x) \in B_H(K)$ *there exists* $y \in I_H(K)$ *such that*
$$\inf_{w \in T(gx)} < w,\psi_2(gx) - \psi_1(y) > \geq f(y) - f(gx).$$

(iv) $fg \geq f\psi_3$ *and* $\psi_2 \cdot g = \psi_1 \cdot \psi_3$.

Then there exists $x_0 \in K$, $w_0 \in T(gx_0)$ *such that* $< w_0,\psi_1(x) - \psi_2(gx_0) > \geq f(gx_0) - f(x)$ *for every* $x \in H$.

Given below are multivalued versions of nonlinear complementarity problems.

Theorem 4.7. *Let* H *be a cone of a locally convex space* E*,* $K \subseteq H$ *compact and convex with* $I_H(K) \neq \phi$, $f : H \to \mathbb{R}^-$ *is a continuous convex function, with* $f(0) = 0$, $T : H \to CK(E^*)$ *and* g, ψ_i $(i = 1,2,3)$ *are maps of* H *into itself leaving* K *invariant with* g *continuous satisfy the following conditions:*

(i) *The maps* $\psi_2 g$ *and* ψ_1 *are continuous where* ψ_1 *is linear and* Tg *uppersemicontinuous,*

(ii) *For every* $x \in K$, *there exists* $w \in T(gx)$ *such that* $< w, \psi_1(x) - \psi_2(gx) > \geq f(gx) - f(x)$.

(iii) *For* $\psi_3(x) \in B_H(K)$ *there exists* $y \in I_H(K)$ *such that*

$$\inf_{w \in T(gx)} < w, \psi_2(gx) - \psi_1(y) > \geq f(y) - f(gx)$$

(iv) $fg \geq f\psi_3$ *and* $\psi_2 \cdot g = \psi_1 \cdot \psi_3$.

(a) *Along with the above conditions if* $f : H \to R^-$ *is such that* $f(\lambda x) = \lambda f(x)$ *for every* $x \in K$ *and* $\lambda \geq 1$, *then there exists* $x_0 \in K$ *and* $w_0 \in T(gx_0)$ *such that* $< w_0, \psi_2(gx_0) > = - f(gx_0)$ *and for every* $x \in H$, $< w_0, \psi_1(x) > \geq 0$.

(b) *Along with the hypotheses (i) - (iv) if* $f : H \to R^-$ *is such that* $f(x+y) \leq f(x)$ *for every* $x,y \in H$, *then there exists* $x_0 \in K$ *and* $w_0 \in T(gx_0)$ *such that* $0 \leq < w_0, \psi_2(gx_0) > \leq - f(gx_0)$ *and* $< w_0, \psi_1(x) > \geq 0$ *for every* $x \in H$.

Proof. By Theorem 4.6 there exists $x_0 \in K$ and $w_0 \in T(gx_0)$ such that

(I) $< w_0, \psi_1(x) - \psi_2(gx_0) > \geq f(gx_0) - f(x)$ for every $x \in H$.

Setting $x = 0$ we get $< w_0, \psi_2(gx_0) > \leq - f(gx_0)$.
In the case (a) taking $x = 2\psi_3(x_0)$ in (I) we have $< w_0, \psi_2(gx_0) > \geq - f(gx_0)$ by using (iv) and $f(\lambda x) = \lambda f(x)$ for $\lambda \geq 1$ and $x \in H$. Thus $< w_0, \psi_2(gx_0) > = - f(gx_0)$ and for $x \in H$, $< w_0, \psi_1(x) > \geq - f(x) \geq 0$ for every $x \in H$.

In case the condition (b) holds, taking $x = 2\psi_3(x_0)$ in (I) and using (iv) and $f(x + y) \leq f(x)$ for every $x,y \in H$ we get $< w_0, \psi_2(gx_0) > \geq 0$. Thus $0 \leq < w_0, \psi_2(gx_0) > \leq - f(gx_0)$. For every $x \in H$, taking $x + \psi_3(x_0)$ in the place of x in (I) and using the conditions on f, ψ_i $(i = 1,2,3)$ it follows that $< w_0, \psi_1(x) > \geq 0$ for every $x \in H$.

Remark 4.3. When g, ψ_i $(i = 1,2,3)$ are chosen as the identity maps on H Theorem 4.7 ((a),(b)) are those obtained in Itoh, Takahashi and Yanagi ([13], Theorems 3.3, 3.4) for continuous f.

When $f \equiv 0$ we get the following result.

Theorem 4.8. *If* H *is a closed convex subset of a locally convex space* E, $K \subseteq H$ *compact and convex with* $I_H(K) \neq \phi$, $T : H \to CK(E^*)$ *and* g, ψ_i $(i = 1,2,3)$ *are maps of* H *into itself leaving* K *invariant with* g *continuous such that the following conditions hold*

(i) *The maps* $\psi_2 g$ *and* ψ_1 *are continuous where* ψ_1 *is linear and* Tg *is uppersemicontinuous*

(ii) *For every* $x \in K$, *there exists* $w \in T(gx)$ *such that* $< w, \psi_1(x) - \psi_2(gx) > \geq 0$

(iii) *For* $\psi_3(x) \in B_H(K)$ *there exists* $y \in I_H(K)$ *such that* $\inf_{w \in T(gx)} < w, \psi_2(gx) - \psi_1(y) > \geq 0$ *and*

(iv) $\psi_2 \cdot g = \psi_1 \cdot \psi_3$.

Then there exists $x_0 \in K$, $w_0 \in T(gx_0)$ *with* $< w_0, \psi_2(gx_0) > = 0$ *and* $< w_0, \psi_1(x) > \geq 0$ *for every* $x \in H$.

The choice $\psi_1 = \psi_2 = g = \psi_3$ in finite dimensional Euclidean space $\mathbb{R}^n$ leads to the following result generalizing the corresponding result due to Itoh, Takahashi and Yanagi [13].

Theorem 4.10. *Let* H *be a cone in the real n dimensional space* $\mathbb{R}^n$ *and* $T : H \to CK(\mathbb{R}^n)$, $g : H \to H$ *a continuous linear map with* $\|g(z)\| \leq \|z\|$ *for every* $z \in H$ *and* Tg *is uppersemicontinuous. Suppose there exists* $c > 0$ *and* $0 \neq v_0 \in T(0)$ *with*

(I) $< w - v_0, g(x) > \geq c\|x\|^2$ *for every* $x \in H$ *and for every* $w \in T(x)$.

If for every $x \in H$, *there exists* $w \in T(gx)$ *such that*

(II) $$< w, gx - g^2x > \geq 0$$

then there exists $x_0 \in H$ *and* $w_0 \in Tg(x_0)$ *such that* $< w_0, g^2(x_0) > = 0$ *and* $< w_0, g(x) > \geq 0$ *for every* $x \in H$.

Proof. If $0 \in T(0) = T(g(0))$ the conclusion holds. Otherwise, define $K = \{x \in H : \|x\| \leq \|v_0\|/c\}$. K is a nonempty compact convex set. Under the given hypotheses, all the conditions of Theorem 4.9 except (iii) are easily verified for the choice $\psi_1 = \psi_2 = g = \psi_3$. If $g(x) \in B_H(K)$ then

$$c\|gx\|^2 = \|v_0\|\|g(x)\| \quad \text{and hence}$$

$$\inf_{w \in T(gx)} < w, g^2x > \geq c\|gx\|^2 + < v_0, g^2(x) > =$$

$$= \|v_0\| \|gx\| + < v_0, g^2(x) >$$

i.e.,

$$\inf_{w \in T(gx)} < w, g^2(x) > \geq 0$$

for the interior point $u = 0$. By Theorem 4.9 there exists $x_0 \in K$ and $w_0 \in T(gx_0)$ such that $< w_0, g^2x_0 > = 0$ and for every $x \in H$ $< w_0, g(x) > \geq 0$.

When g is the identity map on K and $T : H \to CK(\mathbb{R}^n)$ is uppersemicontinuous we obtain the corresponding theorem of Itoh, Takahashi and Yanagi ([13], Theorem 3.6).

The following example illustrates Theorem 4.10.

Example 4.1. Let $H = \mathbb{R}^2_+$ the cone of nonnegative vectors in $\mathbb{R}^2$; $g : H \to H$ be defined by $g(x,y) = (y,x)$ for every $(x,y) \in H$. Define $T : H \to \mathbb{R}^2$ by $T(x,y) = (x^2 + x + y - 1 - \cos x, x^2 + x + y - 1 - \cos x)$. It can be verified that T and g satisfy all the conditions of Theorem 4.10. The set $\{(x,y) : y^2 + y + x - \cos y = 0\} = A$ forms the solutions set, namely for every $(x_0,y_0) \in A$, $< T(g(x_0,y_0)), g^2(x_0,y_0) > = 0$ and for every $(x,y) \in H$, $< T(g(x_0,y_0)), g(x,y) > \geq 0$. $(x_0,y_0) = (2,0)$ is one such solution.

REFERENCES

[1] Allen, G., *Variational inequalities, complementarity problems and duality theorems*, J. Math. Anal. Appl., 58 (1977), 1 - 10.

[2] Baiocchi, C., and Capelo, A., *Variational and quasi-variational inequalities*, John Wiley and Sons, 1984.

[3] Browder, F.E., *Nonlinear monotone operators and convex sets in Banach spaces*, Bull. Amer. Math. Soc., 71 (1965), 780 - 785.

[4] Browder, F.E., *Fixed point theory of multivalued mappings in topological vector spaces*, Math. Ann., 177 (1968), 283 - 301.

[5] Coppoletta, G., *A remark on existence and uniqueness in the theory of variational inequalities*, Boll. Un. Mat. Ital., (5) 16-A (1979), 356 - 359.

[6] Dugundji, J., and Granas, A., *KKM maps and variational inequalities*, Ann. Scuola Norm. Sup. Pisa, (4) 5 (1978), 679 - 682.

[7] Fan, K., *Fixed point and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. USA, 38 (1952), 121 - 126.

[8] Fan, K., *A generalization of Tychonoff's fixed point theorem*, Math. Ann., 142 (1961), 305 - 310.

[9] Fan, K., *Invariant subspaces of certain linear operators*, Bull. Amer. Math. Soc., 69 (1963), 773 - 777.

[10] Fan, K., *Extensions of two fixed point theorems of F.E. Browder*, Math. Z., 112 (1969), 234 - 240.

[11] Gwinner, J., *On fixed points and variational inequalities - A circular tour*, Nonlinear Anal., Theory, Methods and Appl., 5, (5) (1981), 565 - 583.

[12] Iohvidov, I.S., *On a lemma of Ky Fan generalizing the fixed point principle of A.N. Tikhonoff*, Soviet. Math. Dokl., 5 (1964), 1523 - 1526.

[13] Itoh, S., Takahashi, W., Yanagi, K., *Variational inequalities and complementarity problems*, J. Math. Soc. Japan, 30 (1978), 23 - 28.

[14] Jiang Jiahe, *Fixed point theorems for multivalued mappings in locally convex spaces*, Acta Mathematica Sinica, 25 (1982), 366 - 373.

[15] Juberg, R.K., and Karamardian, S., *On variational type inequalities*, *Boll. Un. Math. Ital.*, (4), 7 (1973), 336 - 338.

[16] Karamardian, S., *Generalized complementarity problem*, *J. Optimization Theory Appl.*, 8 (1971), 161 - 168.

[17] Kinderlehrer, D., and Stampacchia, G., *An introduction to variational inequalities and their applications*, Academic Press, New York (1980).

[18] Minty, G.J., *On variational inequalities for monotone operators*, *Adv. Math.*, 30 (1978), 1 - 7.

[19] More´, J.J., *Coercivity conditions in nonlinear complementarity problems*, *SIAM Review*, 16 (1974), 1 - 16.

[20] Mosco, U., *Implicit variational problems and quasi variational inequalities*, *Lecture Notes in Mathematics*, 543: *Nonlinear operators and the calculus of variations*, Bruxelles, 1975, (Editted by J.P. Gossoz, E.L. Dozo, J. Mawhin and L. Waelbroeck, Springer Verlag, Berlin - Heidelberg - New York, 1976), 83 - 156.

[21] Takahashi, W., *Nonlinear variational inequalities and fixed point theorems*, *J. Math. Soc. Japan*, 28 (1976), 168 - 181.

REZIME

JEDNO UOPŠTENJE TEOREME O PRESEKU KY FANA I PRIMENA NA VARIACIONE NEJEDNAKOSTI

U ovom radu su dobijene neke teoreme o preseku. Ove teoreme uopštavaju teoreme o preseku iz [8] i [13]. Date su takodje i neke primene.

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