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ON NUMERICAL SOLUTION OF A SINGULARLY PERTURBED BOUNDARY VALUE PROBLEM II

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ABSTRACT

We sonsider the numerical solution of a nonlinear singularly perturbed two-point boundary value problem by combination of numerical solutions of boundary value problems which approximate the original problem in two parts of the considered interval and the solution of the reduced problem. To approximate the differential equation we use the Hermitian approximation on a special nonuniform mesh. Some numerical examples are given to demonstrate the efficiency of the method.

1. Introduction

In this paper we shall consider the problem

$$T_\varepsilon u := -\varepsilon^2 u'' + c(x,u) = 0, \quad x \in I = [0,1], \qquad (1.1)$$

$$u(0) = u(1) = 0,$$

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where $\varepsilon \in (0,\varepsilon_0)$, $\varepsilon_0 \ll 1$, is a small perturbation parameter. We assume that the following conditions are satisfied:

$$c \in C^k(I \times R), \quad k \in N \tag{1.2}$$

$$g(x) \leq c_u(x,u) \leq G(x), \qquad (x,u) \in I \times R \tag{1.3}$$

$$\delta = \min \{ 5g(x) - 2G(x) : x \in I \} > 0 \tag{1.4}$$

$$0 < \gamma^2 \leq g(x), \quad |g'(x)| \leq L, \quad |G'(x)| \leq L, \quad x \in I \tag{1.5}$$

Numerical treatment of problem (1.1) was considered, among the others, in Bakhvalov [1], Boglaev [2], Doolan, Miller and Schilders [9], Herceg [12], Herceg and Vulanović [13], Marchuk and Schaidurov [20], Shishkin [23] and Vulanović [26] and [28]. The problem (1.1) occurs in the study of chemical catalisys, fluid mechanics (boundary value problems), elasticity, quantum mechanics and fluid dynamics.

We note that the problem (1.1) with conditions (1.2) and

$$0 < \gamma^2 \leq c_u(x,u), \quad (x,u) \in I \times R \tag{1.6}$$

occurs frequently in the literature. In linear case, $c(x,u)=f(x)u + e(x)$, problem (1.1) can be considered under conditions (1.2), with $k \geq 2$, and (1.6) only. Because of $g(x) = G(x) = f(x) \geq \gamma^2$ and $\delta = \min\{3f(x): x \in I\} \geq 3\gamma^2$ conditions (1.3), (1.4) and (1.5) are satisfied.

It is well known that there exists a unique solution $u_\varepsilon \in C^{k+2}(I)$ to (1.1) which in general displays boundary layers at $x=0$ and $x=1$ for small ε, [1],[2],[9],[19]. The corresponding reduced problem $c(x,u)=0$ has also a unique solution $u_0 \in C^k(I)$ which in general does not satisfy the boundary conditions. For the solution u_ε to (1.1) it holds, [23],[25]:

$$|u_\varepsilon^{(i)}(x)| \leq \begin{cases} M(1 + \varepsilon^{-i}\exp(-\gamma x/\varepsilon)), & 0 \leq x \leq 0.5 \\ M(1 + \varepsilon^{-i}\exp(-\gamma(1-x)/\varepsilon)), & 0.5 \leq x \leq 1 \end{cases} \quad i=0,1,\ldots,k. \tag{1.7}$$

Here and throughout the paper M denotes any positive constant that may take different values in different formulas, but that are always independent of ε and of discretization mesh.

In this paper we shall talk about a numerical solution of problem (1.1) which consists of $u_0(x)$ for $x \in [s, 1-s_0]$, s, $s_0 \in (0, 0.5)$ and numerical solutions to problems

$$-\varepsilon^2 v'' + c(x,v) = 0, \quad x \in I = [0,s],$$
$$v(0) = 0, \quad v(s) = u_0(s), \tag{1.8}$$

$$-\varepsilon^2 w'' + c(x,w) = 0, \quad x \in I = [1-s_0, 1],$$
$$w(1-s_0) = u_0(1-s_0), \; w(1)=0. \tag{1.9}$$

A choice of s and s_0 is described in section 2.

From now on we consider a construction of numerical solution to (1.1) on $[0, 0.5]$. A numerical solution on $[0.5, 1]$ can be constructed in a similar way.

Let

$$h := 0.5/n \;, \; I_h = \{x_i = \lambda(a, ih) : i = 0, 1, \ldots, n\}, \tag{1.10}$$

be a special discretization mesh with mesh generating function:

$$\lambda(a,t) = \frac{a\varepsilon t}{q - t}, \; t \in [0, 0.5], \tag{1.11}$$

where

$$q = 0.5 + a\varepsilon \;,$$

and a satisfies

$$0 < 2a\varepsilon < 1. \tag{1.12}$$

Let $I_s = \{ x \in I_h : x \leq s \}$. On I_s we solve (1.8) using Hermitian approximation. If we denote this solution by $v_h = [v_0, v_1, \ldots, v_m]^T$, $m \in N$, $m \leq n$, then we prove that

$$|v(x_i) - v_i| \leq Mh^4 \;, \quad x_i \in I_s$$

where v is the exact solution to (1.8). The existence of v and the following estimates

$$|u_\varepsilon(x) - v(x)| \le M(\exp(-s\gamma/\varepsilon) + \varepsilon^2), \quad x\in [s,0.5], \qquad (1.13)$$

$$|v^{(i)}(x)| \le M(1 + \varepsilon^{-i}\exp(-\gamma x/\varepsilon)), \; 0\le x\le 0.5, \; i=0,1,\dots,k, \qquad (1.14)$$

follow from the inverse monotonicity of (1.8) under assumption (1.3), see [19], [23], [25]. For $s\le x\le 0.5$ we approximate $u_\varepsilon(x)$ by $u_0(x)$ and it holds

$$|u_\varepsilon(x) - u_0(x)| \le M(\exp(-s\gamma/\varepsilon) + \varepsilon^2), \quad x\in [s,0.5]. \qquad (1.15)$$

Using [1.13 - 1.14] we prove

$$|u_\varepsilon(x) - u(x)| \le M(h^4 + \varepsilon^2), \quad x\in I_s \cup [s,0.5],$$

where

$$u(x) = \begin{cases} v_i & \text{for } x=x_i \in I_s, \\ u_0(x) & \text{for } x\in [s,0.5]. \end{cases}$$

For the mesh generating function on [0.5,1] we can take $\lambda_0(a_0,t) = 1 - \lambda(a_0,t)$, $t\in [0.5,1]$, if $0 < 2a_0\varepsilon < 1$. In this case s_0 one can obtain from $s_0=\lambda(a_0,mh)$ where m satisfied $mh < \max\{1-\alpha_0, 1-\alpha_1\} \le (m+1)h$.

Our numerical results are obtained by solving boundary value problems which were cosidered in many papers: [2],[5-11], [15-17],[20],[24],[27-28]. These results show that the theoretical order of converegence is also established numerically.

2. The numerical method

From now we shall take k=8 and a such that (1.12) holds. Our discretization mesh is of the form (1.10) with the mesh generating function λ given by (1.11) and

$$n \geq \max \{3, 3.5L/\delta\} , \tag{2.1}$$

The value of s we choos as follows. Let

$$\alpha_1 = q - qa\varepsilon, \qquad \alpha_0 = q - h(1 + 2\sqrt{3}/3) ,$$

and let $m \in N$ be determined so that

$$(m-1)h \leq \min\{\alpha_0, \alpha_1\} < mh .$$

Then we take

$$s = x_m = \lambda(a, mh) .$$

Since $\alpha_1 < 0.5$ it holds $s \in (0, 0.5)$. Now we shall give some properties of the function $\lambda(a,t)$. It is easily seen that

$$\lambda^{(i)}(t) > 0, \ t \in [0, 0.5], \ i=1,2.$$

Let us concider the following two cases.

C1. $\alpha_0 \leq \alpha_1$. In this case it holds

$$4.5a\varepsilon \leq 9aq\varepsilon \leq (3 + 2\sqrt{3})^2 h^2 , \quad (m-1)h \leq \alpha_0 < mh \leq \alpha_0 + h,$$

and

$$s = \lambda(a, mh) > \lambda(a, \alpha_0) \geq a\varepsilon(3n/(3+2\ 3) - 1) .$$

Thus,

$$\exp(-s\gamma/\varepsilon) \leq \exp(-\lambda(\alpha_0)\gamma/\varepsilon) \leq Mh^4 ,$$

$$\lambda'(a,t) \leq \lambda'(a,mh) \leq \lambda'(a,\alpha_0+h) = 0.75aq\varepsilon/h^2 \leq \sqrt{3}+1.75 , \quad t \in [0, mh].$$

C2. $\alpha_0 > \alpha_1$. Here it holds

$$(m-1)h \leq \alpha_1 < mh \leq \alpha_1 + h \leq \alpha_0 + h,$$

$$s = \lambda(a, mh) > \lambda(a, \alpha_1) = a\varepsilon(\sqrt{1+0.5/(a\varepsilon)} - 1) .$$

Thus,

$$\exp(-s\gamma/\varepsilon) \le \exp(-\lambda(\alpha_1)\gamma/\varepsilon) \le M\varepsilon^2 ,$$

$$\lambda'(a,t) \le \lambda'(a,mh) \le \lambda'(a,\alpha_1+h) \le \lambda'(a,\alpha_0+h) \le \sqrt{3}+1.75$$
$$t \in [0,mh].$$

It follows that in both cases we have

$$\exp(-s\gamma/\varepsilon) \le M(h^2+\varepsilon^2) , \tag{2.2}$$

$$\lambda'(a,t) \le \sqrt{3}+1.75 , \ t \in [0,mh]. \tag{2.3}$$

In order to form a discretization of the problem (1.8) we approximate the differential equation of (1.8) by difference formula of Hermite type in $x_i \in I_s$, $i=1,2,\ldots,m-1$. The coefficients of this formula are not constant, i.e. they depend on x_{i-1}, x_i, x_{i+1}. These coefficients one can obtain in a similar way as on equidistant mesh. Let

$$T^h y_i := a_1(i)y_{i-1} + a_0(i)y_i + a_2(i)y_{i+1} + b_1(i)y''_{i-1} + b_0(i)y''_i + b_2(i)y''_{i+1}$$

where $y_i = y(x_i)$ and $y''_i = y''(x_i)$ for a function $y(x)$.

We obtain the coefficients $a_p(i)$, $b_p(i)$, $p=0,1,2$ from the following system:

$$\begin{aligned} T^h x_i^k &= 0 , \quad k=0,1,2,3,4 \\ b_1(i) + b_0(i) + b_2(i) &= 1. \end{aligned} \tag{2.4}$$

Let $h_i = x_i - x_{i-1}$, $i=1,2,\ldots,n-1$. Then from system (2.4)

$$a_1(i) = \frac{-2}{h_i(h_i+h_{i+1})} , \quad a_2(i) = \frac{-2}{h_{i+1}(h_i+h_{i+1})} , \quad a_0(i) = \frac{2}{h_i h_{i+1}} ,$$

$$b_1(i) = -a_1(i)(h_i^2 - h_{i+1}^2 + h_i h_{i+1})/12,$$

$$b_2(i) = -a_2(i)(h_{i+1}^2 - h_i^2 + h_i h_{i+1})/12,$$

$$b_0(i) = a_0(i)(h_{i+1}^2 + h_i^2 + 3h_i h_{i+1})/12.$$

Using this we approximate the differential equation of (1.8) in $x_i \in I_s$, $i=1,2,\ldots,m-1$, by

$$F_i v_h := \varepsilon^2(a_1(i)v_{i-1} + a_0(i)v_i + a_2(i)v_{i+1}) + b_1(i)c(x_{i-1},v_{i-1}) + b_0(i)c(x_i,v_i) + b_2(i)c(x_{i+1},v_{i+1}) = 0. \tag{2.5}$$

Using this and (2.5) we form discrete analogue of problem (1.8):

$$F_0 v_h := v_0 = 0,$$

$$F_i v_h := \varepsilon^2(a_1(i)v_{i-1} + a_0(i)v_i + a_2(i)v_{i+1}) + b_1(i)c(x_{i-1},v_{i-1}) + b_0(i)c(x_i,v_i) + b_2(i)c(x_{i+1},v_{i+1}) = 0, \quad i=1,2,\ldots,m-1, \tag{2.6}$$

$$F_m v_h := v_m = u_0(s).$$

The solution $v_h=[v_0,v_1,\ldots,v_m]^T$ of (2.6), i.e. of $Fv_h=0$, where $F=(F_0,F_1,\ldots,F_m)$, is the approximation of exact solution u_ε of (1.1) for $x \in I_s$.

Theorem 1. *Suppose that conditions (1.2)–(1.5) are satisfied. Then the equation $Fv_h=0$, i.e. (2.6) has a unique solution v_h which is a point of atraction of SOR-Newton and Newton-SOR methods with relaxation parameter $\omega \in (0,1]$.*

Proof. The Frechet-derivative $F'(z)$ of F for arbitrary $z=[z_0,z_1,\ldots,z_m]^T$ is tridiagonal matrix

$$F'(z) = \begin{bmatrix} 1 & 0 & 0 & & & \\ A_1 & B_1 & C_1 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & A_{m-1} & B_{m-1} & C_{m-1} \\ & & & 0 & 0 & 1 \end{bmatrix},$$

where

$$A_i = \varepsilon^2 a_1(i) + b_1(i) c_u(x_{i-1}, z_{i-1}),$$

$$B_i = \varepsilon^2 a_0(i) + b_0(i) c_u(x_i, z_i),$$

$$C_i = \varepsilon^2 a_2(i) + b_2(i) c_u(x_{i+1}, z_{i+1}).$$

Let

$$\sigma := \min\{ |B_i| - |A_i| - |C_i| : 1 \le i \le m-1 \}.$$

For the coefficients $a_p(i)$, $b_p(i)$, $p=0,1,2$, $i=1,2,\ldots,m-1$ it holds

$$a_1(i) + a_0(i) + a_2(i) = 0, \qquad b_1(i) + b_0(i) + b_2(i) = 1,$$

$$a_1(i) < 0, \quad a_0(i) > 0, \quad a_2(i) < 0, \quad b_0(i) \ge 5/6.$$

Since $i \le m-1$ we have

$$ih \le \alpha_0 \quad \Longleftrightarrow \quad b_1(i) \ge -1/6.$$

For $i=1,2,\ldots,m-1$ it holds

$$-1/6 \le b_1(i) \le b_2(i), \qquad 1/12 \le b_2(i) \le 1/6.$$

Now we consider two cases: $b_1(i) < 0$ and $b_1(i) \ge 0$. In the first case $A_i < 0$ and in dependence of C_i we have the following two subcases.

If $C_i \le 0$ it holds

$$\sigma \geq \varepsilon^2(a_1(i) + a_0(i) + a_2(i)) + b_1(i)c_u(x_{i-1},z_{i-1}) + b_0(i)c_u(x_i,z_i)$$

$$+ b_2(i)c_u(x_{i+1},z_{i+1}) \geq b_0(i)g(x_i) + b_2(i)g(x_{i+1}) + b_1(i)G(x_{i-1})$$

$$\sigma \geq (b_0(i) + b_2(i))g(x_i) + b_1(i)G(x_i) + b_2(i)g'(\theta_{i+1})h_{i+1} + b_1(i)G'(\theta_{i-1})h_i$$

$$6\sigma \geq 11g(x_i) - G(x_i) - |g'(\theta_{i+1})|h_{i+1} - |G'(\theta_{i-1})|h_i \geq \delta - 2Lh_{i+1}$$

$$\sigma \geq \delta/6 - Lh_{i+1}/3.$$

If $C_i > 0$ we have

$$\sigma \geq -2\varepsilon^2 a_2(i) + b_1(i)c_u(x_{i-1},z_{i-1}) + b_0(i)c_u(x_i,z_i) - b_2(i)c_u(x_{i+1},z_{i+1})$$

$$\sigma > b_0(i)g(x_i) + (b_1(i) - b_2(i))G(x_i) - b_2(i)G'(\theta_{i+1})h_{i+1} + b_1(i)G'(\theta_{i-1})h_i$$

$$6\sigma > 5g(x_i) - 2G(x_i) - |G'(\theta_{i+1})|h_{i+1} - |G'(\theta_{i-1})|h_i \geq \delta - 2Lh_{i+1}$$

$$\sigma \geq \delta/6 - Lh_{i+1}/3.$$

In the second case, i.e. in the case $b_1(i) \geq 0$ in a similar way as above we conclude that for $A_i \leq 0$, $C_i \leq 0$ it holds $\sigma \geq \gamma^2$ and in other cases it holds

$$\sigma \geq \delta/6 - Lh_{i+1}/6.$$

It is easy to see that

$$\sigma \geq \min\{\gamma^2,\ \delta/6 - Lh_{i+1}/3\}.$$

Since $h_{i+1} = \lambda'(a,\theta)h$ for some $\theta \in (ih,(i+1)h)$ we have from (2.3)

$$0 < \lambda'(a,t) \leq \sqrt{3}+1.75), \quad t \in [0,mh].$$

Now from (2.1) it follows

$$\sigma \geq \min\{\gamma^2, \delta(1.75 - \sqrt{3}/2)\}$$

and

$$\|F'(z)^{-1}\|_\infty \leq 1/\sigma \leq M. \qquad (2.7)$$

Now by Hadamard Theorem, [21] it follows that equation $Fv_h = 0$ has a unique solution v_h. The matrix $F'(z)$ is srictly diagonaly dominant matrix and convergence of the Newton-SOR and SOR-Newton iterative methods for $\omega \in (0,1]$ follows by well known theorems from [21].

The theorem below is the main result of our paper.

Theorem 3. *Suppose that conditions (1.2)-(1.5) are satisfied. Let u_ε be the solution to problem (1.1) and let*

$$u(x) = \begin{cases} v_i & \text{for} \quad x = x_i \in I_s, \\ u_0(x) & \text{for} \quad x \in [s, 0.5], \end{cases}$$

where $v_h = [v_0, v_1, \ldots, v_m]^T$ is the solution of (2.6) on discretization mesh given by (1.10)-(1.12) with

$$n \geq \max\{3, 3.5L/\delta\}.$$

Then we have

$$|u_\varepsilon(x) - u(x)| \leq M(h^4 + \varepsilon^2), \quad x \in I_s \cup [s, 0.5]. \qquad (2.8)$$

Proof. From (1.15) and (2.2) it follows that (2.8) is valid for $x \in [s, 0.5]$. For $x = x_i \in I_s$ we have

$$|u_\varepsilon(x) - u(x)| \leq |u_\varepsilon(x) - v(x)| + |v(x) - v_i|.$$

Using (2.2) and (1.13) we see that for proof of (2.8) it is sufficient to prove

$$|v^h(x) - v_h(x)| \leq Mh^4, \tag{2.9}$$

where $v^h = [v(x_0),v(x_1),...,v(x_m)]^T$.

For a function $f \in C^6(I)$ we have

$$Rf_i := T^h f_i = (-a_1(i)h_i^5 + a_2(i)h_{i+1}^5)f^{(5)}(x_i)/120$$
$$+ (-b_1(i)h_i^3 + b_2(i)h_{i+1}^3)f^{(5)}(x_i)/6$$
$$+ (a_1(i)h_i^6 + a_2(i)h_{i+1}^6)f^{(6)}(\theta_i)/720$$
$$+ (b_1(i)h_i^4 + b_2(i)h_{i+1}^4)f^{(6)}(\sigma_i)/24 \quad ,$$

with $\theta_i,\sigma_i \in (x_{i-1},x_{i+1})$. Simple calculation shows that

$$Rf_i = (h_{i+1} - h_i)(2h_i^2 + 2h_{i+1}^2 + 5h_ih_{i+1})\ f^{(5)}(x_i)/180$$
$$- (h_i^5 + h_{i+1}^5)f^{(6)}(\theta_i)/(360(h_i + h_{i+1}))$$
$$+ (h_i^4 + h_{i+1}^4 - h_i^2h_{i+1}^2)f^{(6)}(\sigma_i)/144 \ .$$

For the solution v to problem (1.8) we have for $i=1,2,...,m-1$

$$\varepsilon^2T^hv(x_i) = \varepsilon^2Rv(x_i) = \varepsilon^2(a_1(i)v(x_{i-1}) + a_0(i)v(x_i) + a_2(i)v(x_{i+1}))$$
$$+ b_1(i)c(x_{i-1},v(x_{i-1})) + b_0(i)c(x_i,v(x_i))$$
$$+ b_2(i)c(x_{i+1},v(x_{i+1})).$$

It is easy to see that for $i=1,2,...,m-1$

$$\varepsilon^2T^hv(x_i) - F_iv_h = F_iv^h - F_iv_h = \varepsilon^2Rv(x_i).$$

Let $R_h = \varepsilon^2[0,Rv(x_1),Rv(x_2),...,Rv(x_{m-1}),0]^T$. From

$$Fv^h - Fv_h = R_h,$$

it follows

$$F'(z)(v^h - v_h) = R_h,$$

for some $z=[z_0,z_1,...,z_m]^T$. Now from (2.7) we have

$$\|u^h - v_h\|_\infty \le M\|R_h\|_\infty .$$

In order to prove (2.9) we need the estimate

$$\|R_h\|_\infty \le Mh^4$$

i.e.

$$\varepsilon^2|Rv(x_i)| \le Mh^4, \quad i=1,2,...,m-1. \tag{2.10}$$

By using the technique from [25], [25] and [27] we can obtain the last inequality. Here we give the main steps only.

The truncation error of our discretization one can write as

$$\varepsilon^2 Rv(x_i) = \varepsilon^2(P_i + Q_i + S_i),$$

where

$$P_i = (h_{i+1} - h_i)(2h_i^2 + 2h_{i+1}^2 + 5h_ih_{i+1})\ v^{(5)}(x_i)/180,$$

$$Q_i = -(h_i^5 + h_{i+1}^5)v^{(6)}(\theta_i)/(360(h_i + h_{i+1})),$$

$$S_i = (h_i^4 + h_{i+1}^4 - h_i^2h_{i+1}^2)v^{(6)}(\sigma_i)/144 .$$

The estimates of $\varepsilon^2(P_i + Q_i + S_i)$, will be given in the following two cases.

C1. Because of $q>2h$ there is a number $Q \in (1,M]$ such that $n>Q/((Q-1)q)>1/q$. Now let $(i-1)h\le q-Q/(Q-1)n)$. It follows that $(i+1)h<q$ and

$$q - (i-1)h \le Q(q - (i+1)h).$$

Since

$$h_{i+1} - h_i \le Mh^2\lambda''((i+1)h) \ ,$$

$$h_{i+1} = \lambda((i+1)h) - \lambda(ih) \le h\lambda'((i+1)h) \ ,$$

$$\lambda'((i+1)h) = a\varepsilon q(q - (i+1))^{-2} \le a\varepsilon qQ^2(q - (i-1)h)^{-2} \ ,$$

$$\lambda''((i+1)h) = 2a\varepsilon q(q - (i+1))^{-3} \le a\varepsilon qQ^3(q - (i-1)h)^{-3} \ ,$$

$$q - (i-1)h > q - \alpha_1 = \sqrt{a\varepsilon q}$$

we have

$$|P_i| \le Mh^4(a\varepsilon q)^3Q^7(1 + \varepsilon^{-5}\exp(-\gamma\lambda((i-1)h)/\varepsilon))/(q - (i-1)h)^7 \ ,$$

$$|P_i| \le Mh^4\varepsilon^3(1 + \varepsilon^{-5}\exp(-\gamma\lambda((i-1)h)/\varepsilon))/(q - (i-1)h)^7 \ ,$$

$$|P_i| \le Mh^4(\varepsilon^{-1/2} + \varepsilon^{-2}(q - (i-1)h)^{-7}\exp(-\gamma\lambda((i-1)h)/\varepsilon)) \ ,$$

$$\varepsilon^2|P_i| \le Mh^4 \ .$$

Analogously

$$\varepsilon^2|Q_i + S_i| \le Mh^4\varepsilon^2(1 + \varepsilon^{-6}\exp(-\gamma\lambda((i-1)h)/\varepsilon))\lambda'((i+1)h)^4,$$

$$\varepsilon^2|Q_i + S_i| \le Mh^4(1 + \varepsilon^{-4}\exp(-\gamma\lambda((i-1)h)/\varepsilon))\varepsilon^4(q - (i-1)h)^{-8},$$

$$\varepsilon^2|Q_i + S_i| \le Mh^4(1 + (q - (i-1)h)^{-8}\exp(-\gamma\lambda((i-1)h)/\varepsilon)) \le Mh^4 \ .$$

C.2. In this case is $0 < q - 2Qh/(Q-1) < (i-1)h \le \alpha_1$ and $\varepsilon \le M\,h^2$. Now we use

$$\varepsilon^2|Rv(x_i)| \le M\varepsilon^2\max\{|v''(x)| : x\in[x_{i-1},x_{i+1}]\}$$

$$\le M\varepsilon^2(1 + \varepsilon^{-2}\exp(-\gamma\lambda((i-1)h)/\varepsilon)) \le mh^4$$

So in both cases we get (2.10), which completes the proof of the theorem.

3. Numerical examples

In this section we present the results of some numerical experiments using the scheme described in previous section. Our examples are often used in the literature to compare different codes. For some of the selected examples the exact solutions are known. In cases where the exact solution is not available for comparasion purposes, we compute a good approximation to it by asymptotic expansions. We also give the numerical validation of the theoretical order of uniform convergence for the scheme discussed in section 2.

We shall here list our examples and the characteristics of the examples. The boundary conditions for all examples are the same as in (1.1), i.e. $u(0)=u(1)=0$.

Example 1.

Equation: $-\varepsilon^2 u'' + u + \cos^2(\pi x) + 2(\varepsilon\pi)^2\cos(2\pi x) = 0$.

Solution: $u_\varepsilon(x) = (\exp(-x/\varepsilon) + \exp(-(1-x)/\varepsilon))/(1 + \exp(-1/\varepsilon)) - \cos^2(\pi x)$.

References:
- a) Cash, [6]
- b) Dickhoff, Lory, Oberle, Pesh, Rentrop, Seidel, [8]
- c) Deuflhard, Bader, [7]
- d) Doolan, Miller, Schilders, [9]
- e) Hemker, Schippers, de Zeeuw, [11]
- f) Herman, Berndt, [15]
- g) Lentini, Osborne, Russel, [16]
- h) Lentini, Pereyra, [17]
- i) Stoer, Bulirsch, [24]
- j) Vulanović, Herceg, Petrović, [27]
- k) Vulanović, [28] .

Parameters:
$\varepsilon = 1/20$ in b, g, h, i
$\varepsilon = 1/k$, $k=20(20)200$ in c
$\varepsilon = 1/k$, $k=2,10,100,1000$ in e

$\varepsilon = 1/k$, k=100–10 000 in f
$\varepsilon = 10^{-k}$, k=1–8 in a
$\varepsilon = 1/400$ in d
$\varepsilon = 10^{-k}$, k=3,6,9,12 in j,k .

Constants and functions
from (1.3)–(1.5) : $\gamma = 1$, $L = 0$, $g(x)=G(x)=1$, $\delta = 3$.

Condition for n: $n \geq 3$.

Example 2.

Equation: $-\varepsilon^2 u'' + u - 1 = 0$.

Solution: $u_\varepsilon(x) = 1 - ch(x/\varepsilon) - sh(x/\varepsilon)(1 - ch(1/\varepsilon))/sh(1/\varepsilon)$.

References: a) Bohl, [5]
b) Herceg, [14] .

Parameters: $\varepsilon = 1/k$, k=10,20,30,300,1000,100 000 in a
$\varepsilon = 2^{-k}$, k=20–80 in b .

Constants and functions
from (1.3)–(1.5) : $\gamma = 1$, $L = 0$, $g(x)=G(x)=1$, $\delta = 3$.

Condition for n: $n \geq 3$.

Example 3.

Equation: $-\varepsilon^2 u'' + u - 10(2 - \exp(x)) = 0$.

Solution: $u_\varepsilon(x) = 20 - d\exp(x) + (20 - d)(cth(1/\varepsilon)sh(x/\varepsilon) - ch(x/\varepsilon))$
$+ d\exp(1)(20 - d\exp(1))\, sh(x/\varepsilon))/sh(1/\varepsilon)$
$d = 10/(1 - \varepsilon^2)$.

Reference: a) Marchuk, Schaidurov, [20] .

Parameters: $\varepsilon = 1/k$, k=10,20,100 .

Constants and functions from (1.3)-(1.5) : $\gamma = 1$, $L = 0$, $g(x)=G(x)=1$, $\delta = 3$.

Condition for n: $n \geq 3$.

Example 4.

Equation: $-\varepsilon^2 u'' + (1-\varepsilon)u/(1+x)^2 + (1-\varepsilon)x/(1+x)^2 = 0$.

Solution: $u_\varepsilon(x) = -x + (2/(1+x))^{-1/\varepsilon}\,(1-(1+x)^{-2/\varepsilon+1})/(1-2^{-2/\varepsilon+1})$

Reference: a) Boglaev,[2] .

Parameters: $\varepsilon = 1/k$, k=10,100,10 000,1000 000 .

Constants and functions from (1.3)-(1.5) : $\gamma^2 = (1-\varepsilon_0)/2$, $L = 3/4$, $\delta = 2-2.5\varepsilon_0$,
$g(x) = (1-\varepsilon_0)/(1+x^2)$, $G(x)=1/(1+x^2)$

Condition for n: $n \geq \max\{3,\ 10.5/((8-10\varepsilon_0))\}$.

Example 5.

Equation: $-\varepsilon^2 u'' + (1+x)^2 u - (12x^2 - 13x + 5)(1+x)^2 = 0$.

Solution: $u_\varepsilon(x) = 12x^2 - 13x + 5 + 2.5((x/\varepsilon)^2 + x/\varepsilon - 2)\exp(-x/\varepsilon)$
$-4(((x-1)/\varepsilon)^2 - 1.5(x-1)/\varepsilon)\exp(2(x-1)/\varepsilon) + \rho(x)$,
$|\rho(x)| \leq M\varepsilon^2$.

Reference: a) Doolan,Miller,Schilders,[9] .

Parameters: $\varepsilon = 2^{-k}$, k=1(1)9 .

Constants and functions
from (1.3)-(1.5) : $\gamma = 1,\ L = 2,\ g(x)=G(x)=(1+x)^2,\ \delta = 3$.

Condition for n: $n \geq 3$.

Example 6.

Equation: $-\varepsilon^2 u'' + (2x^3 - 3x^2 + 6)u - 4(2x^2 - 3x + 2)((x - .5)^2 + 2) = 0$.

Solution: $u_\varepsilon(x) = 1 - x + (12x^2 - 12x + 8)(x^2 - x + 2.25)/(2x^3 - 3x^2 + 6)$
$- 4\exp(-\sqrt{6}x/\varepsilon) - 3.6\exp(\sqrt{5}(x-1)/\varepsilon) + \rho(x)$,
$|\rho(x)| \leq M\varepsilon^2$.

Reference: a) Doolan, Miller, Schilders, [9] .

Parameters: $\varepsilon = 2^{-k}$, k=1(1)9 .

Constants and functions
from (1.3)-(1.4) : $\gamma^2 = 5,\ L = 3/2,\ \delta = 20$
$g(x)=G(x)= 2x^3 - 3x^2 + 6$.

Condition for n: $n \geq 3$.

Example 7.

Equation: $-\varepsilon^2 u'' + (u - 1)/(1 + d(1 - u)) = 0,\quad 0 \leq d < \sqrt{2.5} - 1$.

Solution: unknown .

References: a) Bohl, [5] .

Parameters: $\varepsilon = 1/k$, k=10,20,30,300,1000,100 000 .

Constants and functions
from (1.3)-(1.5) : $\gamma = 1/(1+d),\quad L = 0,\quad g(x)=1/(1+d)^2$,
$G(x)=1,\quad \delta = 5/(1+d)^2 -2$.

Condition for n: $n \geq 3$.

We denote by E_n the maximum of $|u_\varepsilon(x) - u(x)|$, $x \in I_h$, i.e.

$$E_n = \max\{|u_\varepsilon(x_i) - u(x_i)| : i=0,1,2,\dots,n\} .$$

Also, we define in the usual way the order of convergence Ord for two succsesive values of n with respective errors E_n and E_{2n}:

$$\mathrm{Ord} = (\log(E_n) - \log(E_{2n}))/\log 2 .$$

We expect that Ord = 4 for small ε. Tables 1–3. present the results for numerical solution obtained by our method for examples 1,2 and 6 respectivelly. In table 4 we give some values of α_0 and α_1 as a functions of n and ε.

Table 1. Example 1. a=2.

n \ ε	2^{-4}	2^{-8}	2^{-16}	2^{-32}	2^{-64}–2^{-80}	
4	3 7.92966(-2) –	2 1.44092(-1) –	2 1.35368(-1) –	2 1.35335(-1) –	2 1.35335(-1) –	m E_n Ord
8	6 7.92966(-2) 0.000	6 3.82619(-3) 5.235	6 4.49273(-3) 4.913	6 4.49559(-3) 4.912	6 4.49559(-3) 4.912	
16	12 7.92966(-2) 0.000	15 3.32807(-4) 3.523	14 3.48598(-4) 3.688	14 3.48711(-4) 3.688	14 3.48712(-4) 3.688	
32	23 1.04333(-1) -0.396	29 3.01196(-4) 0.144	30 2.12123(-5) 4.039	30 2.12160(-5) 4.039	30 2.12165(-5) 4.039	
64	45 1.17451(-1) -0.171	57 3.01196(-4) 0.000	62 1.31060(-6) 4.017	62 1.31247(-6) 4.015	62 1.31340(-6) 4.014	
128	89 1.24171(-1) -0.080	114 3.01196(-4) 0.000	126 8.21892(-8) 3.995	126 8.21892(-8) 3.997	126 8.63802(-8) 3.926	

Table 2. Example 2. a=1.

$n \backslash \varepsilon$	2^{-4}	2^{-8}	2^{-16}	2^{-32}	$2^{-64}-2^{-80}$	
4	3 1.35336(-1) –	2 3.73583(-1) –	2 3.67902(-1) –	2 3.67879(-1) –	2 3.67879(-1) –	m E_n Ord
8	7 3.02011(-2) 2.164	6 5.45253(-2) 2.776	6 4.98053(-2) 2.885	6 4.97871(-2) 2.885	6 4.97871(-2) 2.885	
16	13 7.42751(-2) -1.298	14 1.41048(-3) 5.273	14 1.66095(-3) 4.906	14 1.66191(-3) 4.905	14 1.66191(-3) 4.905	
32	25 1.03032(-1) -0.472	30 1.27091(-4) 3.472	30 1.28421(-4) 3.693	30 1.28449(-4) 3.694	30 1.28450(-4) 3.694	
64	49 1.18787(-1) -0.205	59 2.19385(-5) 2.534	62 7.80961(-6) 4.039	62 7.80984(-6) 4.040	62 7.81030(-6) 4.040	
128	97 1.26968(-1) -0.096	118 2.19385(-5) 0.000	126 4.82891(-7) 4.015	126 4.84288(-7) 4.011	126 4.84521(-7) 4.011	

Table 3. Example 6. a=2.

n \ ε	2^{-4}	2^{-8}	2^{-16}	2^{-32}	$2^{-64}-2^{-80}$	
4	3 6.77863(-2) -	2 1.51891(-1) -	2 1.64808(-1) -	2 1.64850(-1) -	2 1.64861(-1) -	m E_n Ord
8	6 5.65400(-3) 3.584	6 9.82175(-3) 3.951	6 1.04836(-2) 3.975	6 1.04861(-2) 3.975	6 1.04861(-2) 3.975	
16	12 8.85083(-3) -0.647	15 5.44772(-4) 4.172	14 6.23719(-4) 4.071	14 6.23875(-4) 4.071	14 6.23876(-4) 4.071	
32	23 9.06107(-3) -0.034	29 4.73624(-5) 3.523	30 3.86313(-5) 4.013	30 3.86424(-5) 4.013	30 3.86387(-5) 4.013	
64	45 9.12677(-3) -0.104	57 4.54336(-5) 0.060	62 2.41213(-6) 4.001	62 2.42237(-6) 3.996	62 2.41585(-6) 3.999	
128	89 9.18783(-3) -0.096	114 4.57680(-5) -0.011	126 1.61119(-7) 3.904	126 1.44355(-7) 4.069	126 1.59256(-7) 3.923	

Table 4. . $\alpha_0 = \alpha_0(n,\varepsilon)$, $\alpha_1 = \alpha_1(\varepsilon)$ for a = 2.

n \ ε	2^{-4}	2^{-8}	2^{-16}	$2^{-32}-2^{-100}$
4	.3556624327	.2384749327	.2306929503	.2306624327
8	.4903312164	.3731437164	.3653617339	.3653312164
16	.5576656082	.4404781082	.4326961258	.4326656082
32	.5913328041	.4741453041	.4663633217	.4663328041
64	.6081664020	.4909789020	.4831969196	.4831664020
128	.6165832010	.4993957010	.4916137186	.4915832010
256	.6207916005	.5036041005	.4958221181	.4957916005
α_1	.3454915028	.4448261113	.4961241484	.4999847417

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REZIME

O NUMERIČKOM REŠAVANJU SINGULARNO PERTURBOVANIH KONTURNIH PROBLEMA II

Posmatra se numeričko rešavanje nelinearnog singularno perturbovanog konturnog problema pomoću kombinacije rešenja odgovarajućeg redukovanog problema i numeričkog rešenja polaznog problema na onom delu intervala koji sadrži granični sloj. Pri tom se za aproksimaciju diferencijalne jednačine koriste Hermitove diferencne formule na specijalnoj neekvidistantnoj mreži. Numerički primeri ilustruju efikasnost predloženog postupka i potvrdjuju teoretske rezultate.

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