ZBORNIK RADOVA
Prirodno-matematičkog fakulteta
Univerziteta u Novom Sadu
Serija za matematiku, 17,1(1987)

REVIEW OF RESEARCH
Faculty of Science
University of Novi Sad
Mathematics Series, 17,1(1987)

REPRESENTATION OF THE SQUARE INTEGRABLE FUNCTIONAL OF THE GAUSSIAN PROCESS WITH THE DISCRETE SPECTRAL TYPE

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ABSTRACT

Let $\{\xi(t)\}$ be a Gaussian process with a discrete spectral type and $H^{(p)}(\xi)$ be the linear closure of Hermite polynomials of degree p in variables $\{\xi(t)\}$. In this paper the innovation process and the spectral type in $H^{(p)}(\xi)$, $p \geq 2$ are determined.

1. INTRODUCTION

Let $\{\xi(t), t > 0\}$, $E\xi(t) = 0$, $E\xi^2(t) < \infty$ be the real mean square continuous and purely non-deterministic process. Denote

$$H_t^{(1)}(\xi) = \bigcap_{\varepsilon>0} \mathcal{L}\{\xi(u),\ u < t+\varepsilon\},$$

where $\mathcal{L}\{\cdot\}$ is the mean square linear closure of random variables in the parenthesis. The space $H^{(1)}(\xi) = \bigcup_t H_t^{(1)}(\xi)$ is a separable Hilbert space with the scalar product $(\xi,\eta) = E\xi\eta$.

The Cramer representation [1] of $\{\xi(t)\}$ is:

AMS Mathematics Subject Classification (1980): 60G15.
Key words and phrases: Gaussian process, spectral type.

(1) $$\xi(t) = \sum_{n=1}^{N} \int_0^t g_n(t,u)\,d\eta_n(u), \quad N \text{ may be } \infty ,$$

where

1. $\{\eta_n(t), t > 0\}$, $n = 1,\dots,N$ are mutually orthogonal increments,

$$E\xi^2(t) = \|\xi(t)\|^2 = \sum_{n=1}^{N} \int_0^t g_n^2(t,u)\,dF_n(u),$$

$$dF_n(u) = d\|\eta_n(u)\|^2,$$

2.

(2) $$\mathcal{H}_t^{(1)}(\xi) = \sum_{n=1}^{N} \oplus\, \mathcal{H}_t^{(1)}(\eta_n)$$

3. The measures dF_n, $n = 1,\dots,N$ are ordered by absolute continuity

$$dF_1 \gtrsim dF_2 \gtrsim \dots \gtrsim dF_N.$$

The correlation function $r(s,t) = E\xi(s)\xi(t)$; $s,t > 0$, uniquely determines the so-called spectral type of $\{\xi(t)\}$ i.e. the chain of equivalence classes of measures

(3) $$\rho_1 \gtrsim \rho_2 \gtrsim \cdots \rho_N,$$

where $dF_n \in \rho_n$.

This fact follows from the well-known theorem on the complete system of unitary invariants of a self-adjoint operator in a separable Hilbert space. Also, one says that (3) is the spectral type of the family $\{\mathcal{H}_t^{(1)}(\xi), t > 0\}$ or of the resolution of the identity $\{P_t, t > 0\}$, where P_t is the projection operator from $\mathcal{H}^{(1)}(\xi)$ onto $\mathcal{H}_t^{(1)}(\xi)$.

The main result of [1] is that for an arbitrary chain (3), there exists a continuous process $\{\xi(t)\}$ with the spectral type (3).

The processes $\{\eta_n(t), t > 0, n = 1,\dots,N\}$ are called the innovation process of $\{\xi(t)\}$ and N is called the multiplicity of $\{\xi(t)\}$. A linear functional ζ of $\{\xi(t)\}$ is an element of $H^{(1)}(\xi)$. It is evident from (1) and (2) that ζ has the following representation

$$\zeta = \sum_{n=1}^{N} \int_0^{\infty} h_n(u)d\eta_n(u), \quad \|\zeta\|^2 = \sum_{n=1}^{N} \int_0^{\infty} h_n^2(u)dF_n(u).$$

2. SQUARE INTEGRABLE FUNCTIONALS OF THE GAUSSIAN PROCESS

In the sequel we suppose that the process $\{\xi(t)\}$ is Gaussian. Consider the set $H(\xi)$ of all square integrable functionals of $\{\xi(t)\}$ i.e. the set of all random variables $X, EX = 0$, $EX^2 < \infty$, measurable with respect to $\{\xi(t)\}$. $H(\xi)$ is a Hilbert space with the scalar product $(X,Y) = EXY$. Since the set of all polynomials of $\xi(t)$, $t > 0$, is dense in $H(\xi)$, it follows that $H(\xi)$ is separable. Denote by $\sigma\{\cdot\}$ the σ-field of random variables in the parenthesis and

$$\mathcal{F}_t(\xi) = \bigcap_{\varepsilon>0} \sigma\{\xi(u), u < t+\varepsilon\}.$$

Let $H_t(\xi)$ be the subspace of $H(\xi)$ consisting of all random variables X, $EX = 0$, $EX^2 < \infty$, measurable with respect to $\mathcal{F}_t(\xi)$. Consider the conditional expectation $E_t(\cdot) = E(\cdot|\mathcal{F}_t(\xi))$. It is evident that $E_tH(\xi) = H_t(\xi)$. It is a well-known fact that $H^{(1)}(\xi)$ reduces E_t to P_t.

In [4] and [5], we solved the problem of the determination of the spectral type of $\{H_t(\xi), t > 0\}$. It is shown there that if ρ_1 in (3) is continuous then the spectral type in $H(\xi)$ is

$$\rho_1 \sim \rho_1 \sim \ldots,$$

i.e. the maximal spectral type ρ_1 in $H^{(1)}(\xi)$ is the uniform maximal spectral type of infinite multiplicity in $H(\xi)$ (terminology of [6]).

The situation is more complicated in the general case ([5]), but ρ_1 is always the maximal spectral type in $H(\xi)$. We shall elaborate the case when ρ_1 is discrete in section 3 of this paper.

In [4] we constructed the innovation process $\{Z_n(t), t > 0,\ n = 1,2,\ldots\}$ in $H(\xi)$ i.e. the mutually orthogonal martingals $\{Z_n(t)\}$, $n = 1,2,\ldots$, satisfying

$$H_t(\xi) = \sum_{n=1}^{\infty} \oplus H_t^{(1)}(Z_n), \quad d\|Z_1(t)\|^2 \gtrsim d\|Z_2(t)\|^2 \gtrsim \ldots$$

The random variable $Z_n(t)$ is expressed as a multiple Itô integral with respect to measures $d\eta_n(t)$, $n = 1,\ldots,N$.

Simply combining the above mentioned main result from [1] and [4], we have

Proposition 1. *There exists a continuous* $\{X(t), t > 0\}$ *such that any square integrable functional* X *of* $\{\xi(t)\}$ *is a linear one of* $\{X(t)\}$

We say that $\{X(t)\}$ *is the process associated to* $\{\xi(t)\}$.

Proof. Let

$$\tau_1 (= \rho_1) \gtrsim \tau_2 \gtrsim \cdots \tag{4}$$

be the spectral type of $\{H_t(\xi)\}$. According to [1], there exists a continuous process $\{X(t), t > 0\}$ such that the chain (4) is its spectral type. It means that

$$H_t^{(1)}(X) = H_t(\xi), \quad H^{(1)}(X) = H(\xi). \tag{5}$$

Let $X \in H(\xi)$. The relation (5) shows that X is the linear functional of $\{X(t)\}$. This completes the proof.

Moreover, since $\{Z_n(t),\ n = 1,2,\ldots\}$ is the innovation process in $H^{(1)}(X)$, the functional X has the representation

$$X = \sum_{n=0}^{\infty} \int_0^{\infty} f_n(n)\,dZ_n(n), \quad \|X\|^2 = \sum_{n=1}^{\infty} \int_0^{\infty} f_n^2(u)\,dG_n(u),$$

$$dG_n(u) \in \tau_n.$$

3. CASE OF THE DISCRETE SPECTRAL TYPE OF $\{\xi(t)\}$

In this section we shall elaborate in detail the construction of the associated process $\{X(t)\}$ and its innovation process when $\{\xi(t)\}$ has the discrete spectral type. The reason of the restriction to the discrete case is that the construction of the process of multiplicity $N > 1$ in the general case involves rather "pathologic" functions $g_n(t,u)$ (see [2]).

Consider the Hermite polynomial $H_p(\xi_1,\ldots,\xi_p)$ of degree p of Gaussian random variables $\xi_1,\ldots,\xi_p$ (not necessarily different). Denote

$$\mathcal{H}_t^{(p)}(\xi) = \bigcap_{\varepsilon>0} \mathcal{L}\{H_p(\xi(u_1),\ldots,\xi(u_p)), u_1,\ldots,u_p < t+\varepsilon\},$$

$$\mathcal{H}^{(p)}(\xi) = \overline{\bigcup_t \mathcal{H}_t^{(p)}(\xi)}.$$

The Hilbert space $\mathcal{H}^{(p)}(\xi)$ reduces $\{E_t\}$, $E_t\mathcal{H}^{(p)}(\xi) = \mathcal{H}_t^{(p)}(\xi)$. Also, there is the orthogonal decomposition

$$\mathcal{H}_t(\xi) = \sum_{p=1}^{\infty} \oplus\, \mathcal{H}_t^{(p)}(\xi). \tag{6}$$

For these reasons it is sufficient to consider the space $\mathcal{H}^{(p)}(\xi)$, $p \geq 2$.

To avoid cumbersome notation, we shall assume that $N = 1$ and that ρ_1 is concentrated on an unbounded sequence of points $0 < t_1 < t_2 < \ldots$. Let

$$\xi(t) = \sum_{t_k \leq t} g_k(t)\eta_k \tag{7}$$

be a Cramér representation. It means that independant Gaussian variables $\eta_1,\eta_2,\ldots$ (say, $E\eta_k^2 = 1$) satisfy $\eta_k \in \mathcal{H}_t^{(1)}(\xi)$ for $t_k \leq t$. To ensure the continuity of $\{\xi(t)\}$, we suppose that $g_k(\cdot)$ is continuous and $g_k(t_k) = 0$, $g_k(t_k+\varepsilon) \neq 0$.

Proposition 2. *The space* $H_t^{(p)}(\xi)$ *coincides with*

$$\mathcal{L}\{H_p(\eta_k,\dots,\eta_{k_p}),\ t_{k_i} \le t\}.$$

Proof. The proof is based on the following property of Hermite polynomials. Consider

$$H_p\Big(\sum_1^{m_1} \xi_i,\ \sum_1^{m_2} \eta_j,\dots,\ \sum_1^{m_p} \zeta_k\Big)$$

where some of the variables ξ_i, η_j, ζ_k may be equal. Examination of the explicit expression of Hermite polynomial (see [3]) yields the relation

$$H_p\Big(\sum_1^{m_1} \xi_i,\ \sum_1^{m_2} \eta_j,\dots,\ \sum_1^{m_p} \zeta_k\Big) = \sum_{i,j,k} H_p(\xi_i,\eta_j,\dots,\zeta_k).$$

We thus conclude that $H_p(\xi(u_1),\dots,\xi(u_p)) \in \mathcal{L}\{H_p(\eta_k,\dots,\eta_{k_p}),$ $t_{k_i} \le max\,u_p\}$. Since (7) is a Cramér representation we have, for some $0 < s_1 \ne s_2 \ne \dots \ne s_k < t_{k+1}$,

$$\eta_k = \sum_{j=1}^{k} a_j \xi(s_j)$$

or $H_p(\eta_{k_1},\dots,\eta_{k_p}) \in H_t^{(p)}(\xi)$ for $t_{k_i} \le t$. This completes the proof.

Lemma. *Two Hermite polynomials of degree* p *in independent variables* $\eta_1,\eta_2,\dots$ *are identical or orthogonal.*

Proof. Since Hermite polynomials are symmetric functions, all $H_p(\eta_{k_1},\dots,\eta_{k_p})$, where $q = (k_1,\dots,k_p)$ is the same combination (with repetition) of $\{1,2,\dots\}$, p at a time, are identical. Let q_j be the number of occurrences of j, j $\in$ $\in \{1,2,\dots\}$ in q. We rewrite

$$H_p(k_1,\dots,k_p) = H_p(q) = H_p(\underbrace{\eta_1,\dots,\eta_1}_{q_1\text{ times}},\underbrace{\eta_2,\dots\eta_2}_{q_2\text{ times}},\dots).$$

By the idenpedence of $\eta_1,\eta_2,\dots$, we have the factorisation

$$H_p(q) = H_{q_1}(\eta_1,\dots,\eta_1)H_{q_2}(\eta_2,\dots,\eta_2)\ \dots\ (H_0(\cdot) = 1).$$

Two combinations q and q′ are different if for at least one j, $q_j \neq q_j'$. So in $EH_p(q)H_p(q')$, one factor is $EH_{q_j}(\eta_j,\ldots,\eta_j)H_{q_j'}(\eta_j,\ldots,\eta_j) = 0$. This completes the proof.

Denote by $C(p,k) = \binom{p+k-1}{p}$, $C(p,0) = 0$, the number of combinations (with repetition) of k elements, p at a time. The are C(p,k) mutually orthogonal Hermite polynomials of degree p in variables $\eta_1,\ldots,\eta_k$. On the figure for p = 3 Hermite polynomials are marked by 0. Observe that $H_t^{(p)}(\xi)$ in the linear closure of C(p,k), $k = \max_{t_j \le t} j$, Hermite polynomials. Hermite polynomials corresponding to the point $t = t_j$ (the number of these is C(p,j) - C(p,j-1) are the innovation received at time $t = t_j$.

Now it is easy to construct the innovation process $\{Z_n^{(p)}(t), t > 0,\ n = 1,2,\ldots\}$

$$Z_1^{(p)}(t) = \sum_{t_j \le t} H_p^{1j},\quad Z_m^{(p)} = \sum_{t_j \le t} H_p^{m_2} (= \sum_{t_1 < t_j \le t} H_p^{m_2}),$$

$$m = 2,\ldots,C(p,2) - C(p,1)$$

$$Z_m^{(p)}(t) = \sum_{t_j \le t} H_p^{m_3} (= \sum_{t_2 < t_j \le t} H_p^{m_3}),$$

$$m = C(p,2) - C(p,1)+1,\ldots,C(p,3) - C(p,2)$$

and so on.

Concerning the spectral type of

$$\{H_t^{(p)}(\xi)\} : d\|Z_1^{(p)}\|^2 \gtrsim d\|Z_2^{(p)}\|^2 \gtrsim \ldots$$

we obtain immediately

Proposition 3. *The spectral type of* $H_t^{(p)}(\xi)$ *is*

$$dF_{11} > dF_{21} \sim dF_{22} \sim \ldots \sim dF_{2,d(p,2)} > dF_{31} \sim$$

$$\sim dF_{32} \sim \ldots \sim dF_{3,d(p,3)} > \ldots$$

where $d(p,k) = [C(p,k) - C(p,k-1)] - [C(p,k-1) - C(p,k-2)]$

and the measure dF_{k_1} *is concentrated at points* $t_k < t_{k+1} < \ldots$

Consider $H(\xi)$. Since (6) holds, $H^{(p)}(\xi)$ reduces $\{E_t\}$ and $d(p,2) = p \to \infty$ as $p \to \infty$, we have

Corollary. *The spectral type of* $\{H_t(\xi)\}$ *is*

$$dF_{11} > dF_{21} \sim dF_{21} \sim \ldots$$

Finally we shall give a Cramér representation of the continuous associated process $\{X^{(p)}(t)\}$ in $H^{(p)}(\xi)$, $p \geq 2$.

Proposition 4. *A continuous process* $\{X^{(p)}(t), t > 0\}$ *with the innovation process* $\{Z_n^{(p)}(t), t > 0,\ n = 1,2,\ldots\}$ *in* $H^{(p)}(\xi)$ *has a Cramér representation*

$$X^{(p)}(t) = \sum_{n\geq 1} \sum_{t_j \leq t} (t - t_j)^j H_p^{nj}$$

(Actually, the domain of the summation of n is $\{1,\ldots,N(t)\}$ where $N(t) = C(p,k) - C(p,k-1)$, $k = \max_{t_j \leq t} j$.

Proof. The continuity of $\{X^{(p)}(t)\}$ follows from

$$\|X^{(p)}(t_j+\varepsilon) - X^{(p)}(t_j-\varepsilon)\|^2 =$$

$$= \sum_{n\geq 1} \|\sum_{i\leq j} (t_j+\varepsilon - t_i)^i H_p^{ni} - \sum_{i\leq i-1} (t_j-\varepsilon-t_i)^i H_p^{ni}\|^2 \leq$$

$$\leq \sum_{n=1}^{N(t)} \{ \sum_{i\leq j-1} [(t_j-t_i+\varepsilon)^i - (t_j-t_i-\varepsilon)^i]^2 \|H_p^{ni}\|^2] +$$

$$+ \varepsilon^{2j} \|H_p^{nj}\|^2\} \to 0, \varepsilon \to 0.$$

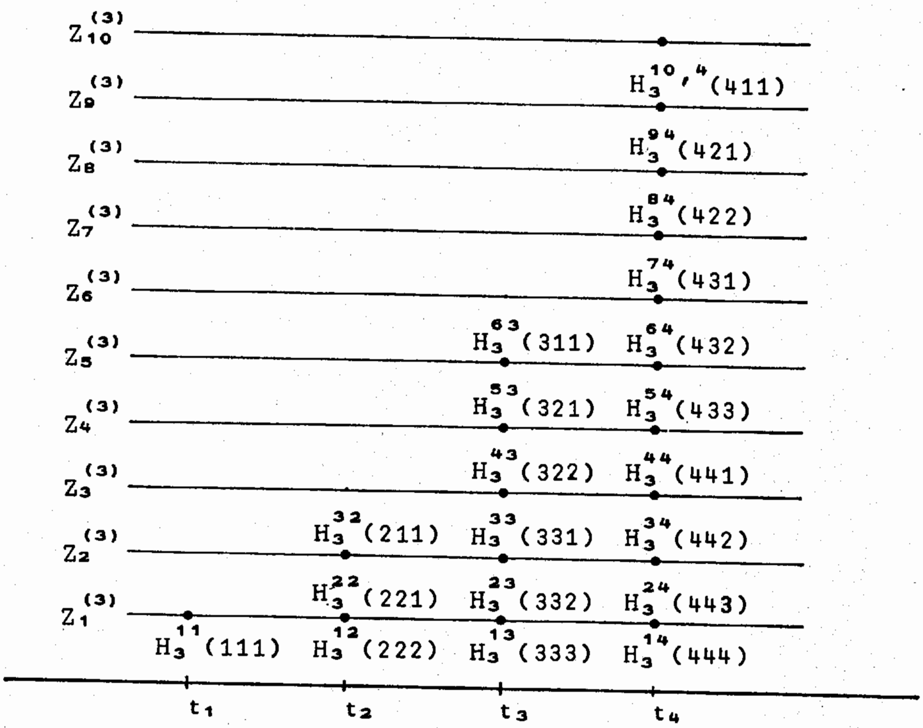
To show that

$$H_t^{(1)}(X^{(p)}) = \sum_{n\geq j} \oplus H_t^{(1)}(Z_n^{(p)})$$

it suffices to show that $H_p^{nj} \in H_t^{(1)}(X^{(p)})$ for $t_j \leq t$. For the sake of simplicity, we shall consider the example $p = 3$ and $t_4 \le t \le t < t_5$, $(k = 4)$. Choose arbitrarily $C(p,k) - C(p,k-1) = 20$ distinct points s_i in $[t_4, t_4+\varepsilon)$, $t_4+\varepsilon < t_5$ and consider the system of 20 linear equations in 20 variables $H_3^{11}, H_3^{12}, \ldots, H_3^{10,4}$:

$$X^{(3)}(s_i) = \sum_{n=1}^{10} \sum_{j\le 4} (s_i - t_j)^j H_3^{nj}, \quad i = 1,\ldots,20.$$

It is not difficult to see that this system has a unique solution, so that H_3^{nj} is a linear combination of $X^{(3)}(s_i)$, $i = 1,\ldots,20$. It means that $H_3^{nj} \in H_t^{(1)}(X^{(3)})$.
This completes the proof.



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REZIME

REPREZENTACIJA KVADRAT-INTEGRABILNIH FUNKCIONALA GAUSOVSKOG PROCESA DISKRETNOG SPEKTRALNOG TIPA

Neka je $\{\xi(t)\}$ Gausovski proces diskretnog spektralnog tipa i neka je $H^{(p)}(\xi)$ linearna zatvorenost Ermitskih polinoma stepena p od promenljivih $\{\xi(t)\}$. U radu se odredjuje inovacioni proces i spektralni tip u $H^{(p)}(\xi)$, $p \geq 2$.

Received by the editors June 1, 1986.