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SOME RESULTS ON BEST APPROXIMATION

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ABSTRACT

Some generalizations of the results from the paper of S.P. Singh [11] are presented.

1. INTRODUCTION

The following result was proved by Brosowski (1969).

Theorem A. *Let* F *be a contractive linear operator on a normed linear space* X. *Let* C *be an* F*-invariant subset of* X *and* x *an* F*-invariant point. If the set of best* C*-approximants to* x *is non-empty, compact and convex, then it contains an* F*-invariant point.*

Later on, it was observed by Singh (1979) that in Theorem A condition of linearity of the operator, and the convexity condition of the set of best approximants can be relaxed. Then, as an application of a fixed point theorem of Edel-

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stein (1962), Singh (1979) obtained the following extension of Theorem A.

Theorem B. *Let* F *be a contractive operator on a normed linear space* X. *Let* C *be an* F*-invariant subsets of* X, *and* x *an* F*-invariant point. If the set of best* C*-approximants to* x *is non-empty, compact and starshaped, then it contains an* F*-invariant point.*

Note. In Theorem A and Theorem B, by a contractive operator $F : X \to X$, it is meant that F satisfies $\|Fx - Fy\| \leq \|x - y\|$ for all x,y in X. However, in the current terminology of fixed point theory, such an operator is said to be non-expansive.

In this paper, we prove some extensions and generalizations of Theorem B by using fixed point theorems of Jungck (1976), Danes (1968) and a notion introduced by Dotson (1973). An application of Theorem B is given which ensures the existence of invariant points for a pair commuting mappings.

2. PRELIMINARIES

We first list a number of definitions and known results for our future use. I denotes the identity mapping in the sequel.

Definition 2.1. (Park (1978)). *Let* G *be a self-mapping on a normed linear space* X. *Then a self-mapping* F *on* X *is said to be* G*-nonexpansive if*

$$(\star) \qquad \|Fx - Fy\| \leq \|Gx - Gy\|$$

for all $x,y \in X$.

If strict inequality holds in $(\star)$ for distinct points, then F is said to be "G-contractive". For $G = I$, we simply say that F is

"contractive". Clearly, G-nonexpansive and G-contractive maps are continuous, whenever G itself is a continuous map.
nuous map.

Jungck (1976) obtained the following common fixed point theorem for G-contractive mappings.

Theorem C. *Let* F *and* G *be commuting mappings of a compact metric space* (X,d) *into itself such that* $F(X) \subset G(X)$, *and* G *is continuous. If* F *is a* G-*contractive map on the metric space* X, *then there is a unique common fixed point of* F *and* G.

A result similar to Theorem C for contractive mappings has been proved by Edelstein (1962).

Definition 2.2. *A subset* M *of a linear space* X *is said to be "starshaped" provided that there is at least one* $p \in M$ *such that, if* $x \in M$ *and* $0 \leq \alpha < 1$, *then* $(1-\alpha)p + \alpha x \in M$.

Each convex set is necessarily starshaped, but a starshaped set need not be convex.

Definition 2.3. (**Dotson (1973)**). *Let* S *be a subset of a normed linear space* X, *and let* $H = \{\phi_i\}_{i \in S}$ *be a family of functions from* [0,1] *into* S, *having the property that for each* $i \in S$, *we have* $\phi_i(1) = i$. *Such a family* H *is said to be "contractive" provided there exists a function* $\theta : (0,1) \to (0,1)$ *such that for all* i *and* j *in* S *and for all* $t \in (0,1)$ *we have*

$$\|\phi_i(t) - \phi_j(t)\| \leq \theta(t)\|i - j\|.$$

Such a family H *is said to be "jointly continuous" provided that if* $t \to t_0$ *in* [0,1] *and* $i \to i_0$ *in* S *then* $\phi_i(t) \to \phi_{i_0}(t_0)$ *in* S.

It may be remarked that if S is a starshaped subset of a normed linear space then there exists a contractive jointly continuous family of functions associated with S as described above.

Now we introduce the following:

Definition 2.4. *A subset* M *of a linear space* X *is said to be "G-starshaped" with respect to a mapping* $G : X \to X$ *provided there is at least one* $p \in M$ *such that if* $x \in M$ *and* $\alpha \in (0,1)$ *then* $(1-\alpha)Gp + \alpha x \in M$.

3. MAIN RESULTS

The first theorem generalizes the main result of Singh (1979).

Theorem 3.1. *Let* F *and* G *be commuting operators on a normed linear space* X *such that* F *is* G*-nonexpansive, where* G *is linear, continuous and satisfies* $G^2 = G$. *Let* C *be a subset of* X, *and* x *a point of* X *such that both of them are invariant under both* F *and* G.

Let $D = \{y \in C : Gy$ *is a best* C*-approximant to* $x\}$. *If* $F(D) \subset G(D)$, *and also* D *is non-empty, compact and* G*-starshaped with respect to* G, *then* D *contains a point invariant under both* F *and* G.

Proof. Let $y \in D$. Then we have

$$\|GFy - x\| = \|FGy - Fx\| \leq \|G^2y - Gx\| = \|Gy - x\|,$$

and

$$\|G(Gy) - x\| = \|Gy - x\|.$$

These relations show that $Fy \in D$ and $Gy \in D$. Thus F and G are self-mappings on D.

Let $p \in D$ such that $\alpha q + (1-\alpha)Gp \in D$ for all $q \in D$, and $0 \leq \alpha < 1$. Let $\{\alpha_n\}$ be a sequence of real numbers such that

$0 \leq \alpha_n < 1$ and $\lim_{n\to\infty} \alpha_n = 1$. Now define $F_n : D \to D$ by setting

$$F_n x = \alpha_n Fx + (1-\alpha_n)Gp$$

for all $x \in D$. For each n, F_n is clearly a self-mapping on D. Now

$$\begin{aligned} \| F_n x - F_n y\| &= \alpha_n \| Fx - Fy\| \\ &\leq \alpha_n \| Gx - Gy\| \\ &< \| Gx - Gy\| , \end{aligned}$$

for all $x,y \in D$, $Gx \neq Gy$.
Using commutativity of G and F, linearity of G and the fact $G^2 = G$, it is routine to verify that G and F_n commutes for each n. Also because of $F(D) \subset G(D)$, linearity of G and $G^2 = G$, it is not difficult to see that $F_n(D) \subset G(D)$ for all n. As D is compact , it follows from Theorem C that there is a unique common fixed point, say $x_n \in D$, of F_n and G for each n. So $Gx_n = x_n = F_n x_n$. Once again the compactness of D ensures that $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ converging to an $\bar{x} \in D$. We claim that $\bar{x} = F\bar{x} = G\bar{x}$. To show this, consider

$$x_{n_i} = F_{n_i} x_{n_i} = \alpha_{n_i} F x_{n_i} + (1-\alpha_{n_i})Gp.$$

Letting $i \to \infty$ and using the continuity of F, we find that $\bar{x} = F\bar{x}$. Furthermore, $G\bar{x} = G(\lim_{i\to\infty} x_{n_i}) = \lim_{i\to\infty} Gx_{n_i} = \lim_{i\to\infty} x_{n_i} = \bar{x}$. Thus $\bar{x}$ is the simultaneous invariant point of F and G as required.

Remark. It appears from the proof of Theorem 3.1. that the continuity and the linearity of G restricted to D only are actually needed.
Next result is yet another extension of Theorem B, where we

use Theorem 4 of Danes (1968) instead of Edelstein's theorem as used by Singh (1979).

Theorem 3.2. *Let F be a contractive weakly continuous operator on a normed linear space X. Let C be a F-invariant subset of X and x a F-invariant point. If the set of best C-approximants to x is non-empty, weakly compact and star-shaped, then it contains a F-invariant point.*

The following theorem of Meade and Singh (1977) extends a result of Kasahara (1976)

Theorem D. *Let G be a mapping of a set X into itself such that the function $x \mapsto d(Gx, G^2x)$ has a minimum value at some $a \in X$, where d is a non-negative real-valued function on $X \times X$ such that $d(x,y) = 0$ if and only if $x = y$. Then G has a fixed point if there is a mapping F of X into G(X) which commutes with G and satisfies (☆) in X for all $x,y \in X$ with $Gx \neq Gy$.*

Let Y be a subset of a Banach space X. Then we shall say that Y has property (MS) with respect to G if there exists a mapping $G : Y \to Y$ such that the function $x \mapsto d(Gx, G^2x)$ has a minimum value at some $a \in Y$, where d is as in Theorem D.

Clearly, a compact subset Y of X will always have property (MS) with respect to a continuous function G. From this we get the following slightly revised version of Theorem 3.1.

Theorem 3.3. *Let F and G be commuting operators on a normed linear space X such that F is G-nonexpansive, where G is linear, continuous and satisfies $G^2 = G$. Let C be a subset of X and x a point of X such that both of them are invariant under F and G.*
Let $D = \{y \in C : Gy$ is best C-approximant to $x\}$. If $F(D) \subset G(D)$, and also D is non-empty, G-starshaped and has property (MS) with respect to G, then D contains a point invariant under both

F *and* G.

Proof. The proof is almost identical to that of Theorem 3.1 except that we use Theorem D rather than Theorem C.

Finally, we extend the main theorem of Singh (1979) for a single operator where the starshaped sets are replaced by sets with which a certain family of functions are associated.

Theorem 3.4. *Let* F *be a nonexpansive operator on a normed linear space* X. *Let* C *be an* F*-invariant subset of* X *and* x *an* F*-invariant point. If the set of best* C*-approximants to* x *is non-empty, compact and for which there exists a contractive jointly continuous family* H *of functions, then it contains an* F*-invariant point.*

Proof. Let D be the set of best C-approximants to x. Then F is clearly a self-mapping on D. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\lim_{n\to\infty} \alpha_n = 1$. Let us define $F_n : D \to D$ by

$$F_n(x) = \phi_{Fx}(\alpha_n)$$

for each $x \in D$. Since $\alpha_n \in (0,1)$ and F is a self-mapping on D, we see that F_n is well-defined for each n and is also a self-mapping on D.

Now for all $x,y \in D$ and each n we have

$$\|F_n x - F_n y\| = \|\phi_{Fx}(\alpha_n) - \phi_{Fy}(\alpha_n)\| \leq \theta(\alpha_n)\|Fx - Fy\|$$

$$\leq \theta(\alpha_n)\|x - y\| < \|x-y\|$$

for all $x,y \in D$, $x \neq y$.

Therefore each F_n is a contractive mapping on D. Then, by a fixed point theorem of Edelstein (1962), there exists for each n, a unique point $x_n \in D$ such that $F_n(x_n) = x_n$. The rest of the proof is identical to that of the proof of Theorem 3.1.where we make use of compactness of D, continuity of F, the joint continuity of the family H and the Hausdorffness of the space. This ends the proof.

4. AN APPLICATION

The next result is a direct application of the main theorem of Singh (1979). In doing so, we are motivated by the work of Goebel and Zlotkiewicz (1971) where some fixed point theorems in Banach spaces have been obtained for self-mappings G which are non-expansive and also satisfy $G^2 = I$.

Theorem 4.1. *Let* F *and* G *be commuting operators on a normed linear space* X *such that* F *is* G-*nonexpansive and* $G^2 = I$

Let C *be an* FG-*invariant subset of* X *and* x *an* FG-*invariant point of* X. *If the set* D *of best* C-*approximant to* x *is a singleton set* $\bar{x}$, *then* $\bar{x}$ *is a simultaneous invariant point of* F *and* G.

Proof. As $G^2 = I$, FG is clearly a nonexpansive mapping. Then by Theorem B, the set D of best C-approximant to x contains an FG-invariant point which clearly coincides with $\bar{x}$. Now consider

$$FG(F\bar{x}) = F(GF\bar{x}) = F(FG\bar{x}) = F\bar{x}.$$

Then by unicity of $\bar{x}$, we get $F\bar{x} = \bar{x}$. Furthermore, $G(\bar{x}) = G(F\bar{x}) = GF\bar{x} = \bar{x}$. Thus $\bar{x}$ is a point which is invariant under both F and G, as desired.

Remarks (i) It may be noted that no continuity condition either on F or on G is required.

(ii) It is well-known (Hicks and Humphries (1982)) that in a strictly convex normed linear space, the set of best approximants contains only one element.

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REZIME

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