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A NOTE ON SUBSETS AND ALMOST CLOSED MAPPINGS

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ABSTRACT

The aim of the present paper is to study some properties of α-Hausdorff (α-regular, α-almost regular) subsets and almost closed mappings.

1. INTRODUCTION

Our notation is standard. No separation properties are assumed for spaces unless explicitly stated.

A subset A of a space X is <u>regularly open</u> iff it is the interior of its own closure, or equivalently, iff it is interior of some closed set. A is called <u>regularly closed</u> iff it is the closure of some open set, or equivalently, iff it is

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a closure of its own interior (a subset is regularly open iff its complement is regularly closed), [6].

A mapping $f: X \to Y$ is said to be almost closed (almost open) iff for every regularly closed (regularly open) set F of X, f(F) is closed (open) in Y, [6].

Let X be a space and A a subset of X. The set A is α-paracompact (α-nearly paracompact) iff for every X-open (X-regularly open) cover $\mathcal{U}$ of A there exists an X-open X-locally finite family $\mathcal{V}$ which refines $\mathcal{U}$ and covers A, [8]([3]).

A subset A of a space X is α-nearly compact (N closed) iff every X-regularly open cover of A has a finite subcover, [7].

A subset A of a space X is Lindelöf iff every X-open cover of A has a countable subcover, [1].

A subset A of a space X is α-Hausdorff iff any two points a,b of a space X, where $a \in A$ and $b \in X \setminus A$, can be strongly separated, [4].

A subset A of a space X is α-regular (α-almost regular) iff for any point $a \in A$ and any open (regularly open) set U containing a, there exists an open set V such that $a \in V \subset \bar{V} \subset U$, [4]([2]).

2. RESULTS

LEMMA 2.1. *Let* $\mathcal{U} = \{U_i : i \in I\}$ *be any family of α-regular (α-almost regular, α-Hausdorff) subsets of a space X. Then the sets* $U = \cup\{U_i : i \in I\}$ *and* $V = \cap\{U_i : i \in I\}$ *are α-regular (α-almost regular, α-Hausdorff).*

PROOF. Obvious.

In paper [4]([2]) the author showed that if A is an α-regular (α-almost regular) α-paracompact (α-nearly paracompact) subset of a space X, U an open (regularly open) neighbourhood of A, then there exists an open (regularly open) neighbourhood V of A such that $A \subset V \subset \bar{V} \subset U$.

From this fact we can easily prove the next lemma:

LEMMA 2.2. *Let* A *be any open (regularly open)* α*-regular* α*-paracompact (*α*-almost regular* α*-nearly paracompact) subset of a space* X. *Then* A *is closed,(*A *is regularly closed).*

THEOREM 2.1. *In any space the union of a locally finite family of open* α*-paracompact* α*-regular sets is a clo--open* α*-regular* α*-paracompact set.*

PROOF. Let $\mathcal{U} = \{U_i : i \in I\}$ be any locally finite family of open α-regular α-paracompact subsets of a space X. By theorem 9 in [8], the set $U = \cup\{U_i : i \in I\}$ is α-paracompact. By Lemma 2.1. U is α-regular. By Lemma 2.2. the set U is closed.

THEOREM 2.2. *Let* A *and* B *be any disjoint closed* α*-regular and Lindelöf subsets of a space* X. *Then, there exist disjoint open sets* U *and* V *such that* $A \subset U$, $B \subset V$.

PROOF. For each point $x \in A$ there exists an open set U_x, such that $x \in U_x \subset \bar{U}_x \subset X \setminus B$. For each point $x \in B$ there exists an open set V_x, such that $x \in V_x \subset \bar{V}_x \subset X \setminus A$. Let $\mathcal{U} = \{U_x : x \in A\}$. Let $\mathcal{V} = \{V_x : x \in B\}$. Since A is Lindelöf there exists a sequence $\{U_n : n \in \omega\}$ of elements of a family $\mathcal{U}$, such that $A \subset \cup\{U_n : n \in \omega\}$. Since B is Lindelöf, there exists a sequence $\{V_n : n \in \omega\}$ of elements of a family $\mathcal{V}$, such that $B \subset \cup\{V_n : n \in \omega\}$. Let $U'_n = U_n \setminus \cup\{\bar{V}_p : p \leq n\}$ and $V'_n = V_n \setminus \cup\{\bar{U}_p : p \leq n\}$. Since $U'_n \cap V'_m = \emptyset$ for each n and m, it follows that $A \subset U = \cup\{U'_n : n \in \omega\}$, $B \subset V = \cup\{V'_n : n \in \omega\}$ and $U \cap V = \emptyset$.

COLOLLARY 2.1. *Let* A *and* B *be any disjoint closed* α*-regular subsets of a Lindelöf space* X. *Then, there exist disjoint open sets* U *and* V *such that* $A \subset U$, $B \subset V$.

PROOF. Every closed subset of a Lindelöf space is Lindelöf.

In paper [5] T.Noiri proved the next theorem:

THEOREM A. *If* $f: X \to Y$ *is an almost closed mapping of a Hausdorff space* X *onto a compact space* Y *with* N-*closed point inverses, then* f *is continuous.*

In this theorem the Hausdorff property can be omitted, which we shall prove in the next theorem.

THEOREM 2.3. *If* $f: X \to Y$ *is an almost closed mapping of a space* X *onto a compact space* Y *such that* $f^{-1}(y)$ *is* α-*Hausdorff* α-*nearly paracompact for each point* $y \in Y$, *then* f *is continuous.*

PROOF. Suppose that f is not continuous at some point $x \in X$. Let $\mathcal{U}(x)$ denote the family of all open neighbourhoods of x in X. Let $y = f(x)$. Since f is not continuous at x, then there exists an open neighbourhood V of y in Y such that $f(U) \cap (Y \setminus V) \neq \emptyset$ for every $U \in \mathcal{U}(x)$. Thus $A = \{f(\bar{U}) \cap (Y \setminus V): U \in \mathcal{U}(x)\}$ is a family of closed subsets of Y. This family must have the finite intersection property (if there exists a finite number of open sets $U_1, U_2, \ldots, U_n$ such that $\bigcap_{i=1}^{n} [f(\bar{U}_i) \cap (Y \setminus V)] = \emptyset$, then $\bigcap_{i=1}^{n} U_i$ is an open set containing x and $(Y \setminus V) \cap f(\bigcap_{i=1}^{n} U_i) \subset$ $\subset \bigcap_{i=1}^{n} (Y \setminus M) \cap (f(\bar{U}_i)) = \emptyset$ which is a contradiction).
Since Y is compact, there exists a point $y_o \in \cap\{A: A \in A\}$. Thus we have $y_o \in Y \setminus V$ and hence $x \notin f^{-1}(y_o)$.
Since $f^{-1}(y_o)$ α-Hausdorff α-nearly paracompact, it follows that, by Lemma 2.1 in [2], there exist disjoint regularly open sets U_x and U_o such that $x \in U_x$ and $f^{-1}(y_o) \subset U_o$.
Since $\bar{U}_x \cap f^{-1}(y_o) \subset \bar{U}_x \cap U_o = \emptyset$ we have $y_o \notin f(\bar{U}_x)$.
On the other hand, since U_x belongs to $\mathcal{U}(x)$, we have $y_o \in f(\bar{U}_x) \cap (Y \setminus V) \subset f(\bar{U}_x)$. This is a contradiction. Hence

f must be continuous at x. Since x is an arbitrary point of X, it follows that f is continuous.

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REZIME

O PODSKUPOVIMA I SKORO ZATVORENIM PRESLIKAVANJIMA

U radu se ispituju neke osobine α-Hausdorfovih, α-regularnih i α-skoro regularnih podskupova topološkog prostora X. Daju se i uslovi kada je skoro zatvoreno preslikavanje neprekidno nad prostorom koji ne mora da bude Hausdorfov.

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