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ON α-PRODUCTS OF DISTRIBUTIONS

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ABSTRACT

Using the neutrix calculus we introduce and analyse a more general product of distributions than in [4]. In particular, we find the "α-product" $x^{\lambda} \overset{\alpha}{\circ} x_-^{\mu}$ for $\lambda + \mu < -1$, $\lambda, \mu, \lambda+\mu \neq -1, -2, \ldots$ and $\alpha = -\lambda-\mu-1-[-\lambda-\mu]$.

☆

In the following ρ denotes a fixed infinitely differentiable function having the properties

(i) $\rho(x) = 0$ for $|x| \geqq 1$,

(ii) $\rho(x) \geqq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)\,dx = 1$.

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The function δ_n is defined by $\delta_n(x) = n\rho(nx)$ for $n = 1,2, \ldots$. It is obvious that the sequence $\{\delta_n\}$ is regular and converges to the Dirac delta-function δ.

The following definitions and theorem were given in [4].

Definition 1. Let h_r be a distribution for $r = 0,1,2, \ldots$. We say that

$$\underline{h} = [h_0,h_1, \ldots ,h_r, \ldots]$$

is a distribution vector.

If $h_{r+i} = 0$ for $i = 1,2, \ldots$, we write

$$\underline{h} = [h_0,h_1, \ldots ,h_r,0, \ldots] = [h_0,h_1, \ldots ,h_r]$$

and if $h_i = 0$ for $i = 1,2, \ldots$, we write

$$\underline{h} = [h_0] = h_0.$$

The set of all distribution vectors is made into a vector space by defining the sum and product by a scalar in the usual way.

Definition 2. Let $\underline{h} = [h_0,h_1, \ldots ,h_r, \ldots]$ be a distribution vector and let ϕ be an arbitrary test function with compact support. We define $(\underline{h}, \phi)$ by the sequence of real numbers

$$(\underline{h}, \phi) = ((h_0, \phi),(h_1, \phi), \ldots ,(h_r, \phi), \ldots).$$

Definition 3. Let $\underline{h} = [h_0,h_1, \ldots ,h_r, \ldots]$ be a distribution vector. We define the derivative $\underline{h}'$ of $\underline{h}$ by

$$\underline{h}' = [h_0',h_1', \ldots ,h_r', \ldots].$$

Theorem 1. Let $\underline{h} = [h_0,h_1, \ldots ,h_r, \ldots]$ be a distribution vector and let ϕ be an arbitrary test function with compact support. Then

$$(\underline{h}', \phi) = -(\underline{h}, \phi').$$

Definition 4. Let f and g be distributions and let $g_n = g * \delta_n$. We say that the neutrix product $f \circ g$ of f and g exists and is equal to the distribution vector $\underline{h} = [h_0, h_1, \ldots, h_r, \ldots]$ on the open interval (a,b) if

$$\underset{n\to\infty}{\text{N-lim}}\, n^{-r}(f, g_n\phi) = (h_r, \phi)$$

for $r = 0,1,2, \ldots$ and all test functions ϕ in $D(a,b)$, where N is the neutrix with negligible functions, linear sums of the functions $n^{\lambda}\ln^{r-1}n$, $\ln^r n$ for $\lambda > 0$ and $r = 1,2, \ldots$ and all functions which converge to zero as n tends to infinity.

This definition of the neutrix product was introduced in order to give more information about the behaviour of the neutrix product than was given by definition 4 of [2]. Although this is indeed so for a number of important neutrix products, it fails for other neutrix products.

In order to remedy this we have

Definition 5. Let f and g be distributions and let $g_n = g * \delta_n$. We say that the <u>α-neutrix product</u> $f \overset{\alpha}{\circ} g$ of f and g exists and is equal to the distribution vector $\underline{h} = [h_0, h_1, \ldots, h_r, \ldots]$ on the open interval (a,b) if

$$\underset{n\to\infty}{\text{N-lim}}\, (f, g_n\phi) = (h_0, \phi),$$

$$\underset{n\to\infty}{\text{N-lim}}\, n^{-\alpha-r}(f, g_n\phi) = (h_r, \phi)$$

for $r = 1,2, \ldots$ and all test functions ϕ in $D(a,b)$, where

$-1 < \alpha \leq 0.$

It is immediately obvious that definition 5 is equivalent to definition 4 in the particular case $\alpha = 0$.

Definition 6. Let f and g be distributions and suppose that the α-neutrix product $f \overset{\alpha}{\circ} g$ exists and is equal to the distribution vector $\underline{h} = [h_0, h_1, \ldots, h_r, \ldots]$ on the open interval (a,b). We say that h_0 is the finite part of $f \overset{\alpha}{\circ} g$ and then write

$$\text{p.f. } f \overset{\alpha}{\circ} g = h_0$$

on the interval (a,b).

It is obvious that by taking the finite part of an α-neutrix product reduces definition 5 to the original definition of the neutrix product, see definition 4 of [2].

Theorem 2. Let f and g be distributions and suppose that the α-neutrix products $f \overset{\alpha}{\circ} g$ and $f' \overset{\alpha}{\circ} g$ (or $f \overset{\alpha}{\circ} g'$) exist and are equal to distribution vectors on the open interval (a,b). Then the α-neutrix product $f \overset{\alpha}{\circ} g'$ (or $f' \overset{\alpha}{\circ} g$) exists as a distribution vector and

$$(f \overset{\alpha}{\circ} g)' = f' \overset{\alpha}{\circ} g + f \overset{\alpha}{\circ} g'$$

on the interval (a,b).

We omit the proof of this theorem as it is almost identical to the proof of theorem 2 of [3].

Theorem 3. Let f and g be tempered distributions such that the α-neutrix product $f \overset{\alpha}{\circ} g$ exists and is equal to $\underline{h}$ on the real line. Then $\underline{h}$ has only finitely many non-zero terms.

Proof. It is well-known, (see [6], theorem 12, p. 41), that there exist integers k, k', l and l' and continuous functions F and G on the real line such that

$$(1) \qquad f = F^{(k)}, \quad g = G^{(k')}$$

and

$$(2) \quad |F(x)| \leqq K(1+|x|)^{l}, \quad |G(x)| \leqq K(1+|x|)^{l'}$$

for some $K > 0$ and all real x.

Then (1) implies for arbitrary test function ϕ with compact support

$$(f, g_n\phi) = (-1)^k(F, (g_n\phi)^{(k)})$$

$$= (-1)^k \sum_{j=0}^{k} \binom{k}{j} n^{k'+j} \int_{-\infty}^{\infty} F(x)\phi^{(k-j)}(x) \int_{-1}^{1} G(x-t/n)\rho^{(k'+j)}(t)\,dt\,dx.$$

It now follows from (2) that

$$|(f, g_n\phi)| \leqq$$

$$k!n^{k+k'} \sum_{j=0}^{k} \int_{-\infty}^{\infty} (1+|x|^{l+l'}|\phi^{(k-j)}(x)|\,dx . \int_{-1}^{1} (1+|t|)^{l'}|\rho^{(k'+j)}(t)|\,dt$$

$$= O(n^{k+k'}).$$

Thus

$$|n^{-r}(f, g_n\phi)| = O(n^{-r+k+k'})$$

and so

$$\underset{n\to\infty}{\text{N-lim}}\; n^{-r}(f, g_n\phi) = \lim_{n\to\infty} n^{-r}(f, g_n\phi) = 0$$

for $r > k + k'$. The result of the theorem follows.

The proof of this theorem can be modified to give

Theorem 4. Let f and g be distributions such that the α-neutrix product

$f \overset{\alpha}{\circ} g$ exists and is equal to $\underline{h}$ on the finite open interval (a,b). Then $\underline{h}$ has only finitely many non-zero terms.

Definition 7. Let f and g be distributions and let $g_n = g * \delta_n$. We say that the <u>α-product</u> $f \overset{\alpha}{\circ} g$ of f and g exists and is equal to the distribution vector $\underline{h} = [h_0, h_1, \ldots, h_r, \ldots]$ on the open interval (a,b) if

$$(f, g_n\phi) = (h_0, \phi) + \sum_{r=1}^{\infty} (h_r, \phi) n^{\alpha+r} + \varepsilon_n$$

where $\varepsilon_n = O(n^\alpha)$ if $-1 < \alpha < 0$ and $\varepsilon_n \to 0$ if $\alpha = 0$. for all test functions ϕ in $D(a,b)$.

In particular if $h_r = 0$ for $r = 1,2, \ldots$ we simply say that the product $f \circ g$ of f and g exists and write

$$f \circ g = h_0$$

on the interval (a,b).

It follows that this definition of the product $f \circ g$ is equivalent to definition 4 of the product $f \circ g$ given in [1].

Further, we note that if the product $f \circ g$ exists and equals h_0, then the α-product $f \overset{\alpha}{\circ} g$ exists and is equal to h_0 for all α with $-1 < \alpha \leq 0$.

We also note that if the α-product of two distributions f and g exists and is equal to the distribution vector $\underline{h}$, then the α-neutrix product of f and g exists and is equal to $\underline{h}$, although the converse does not hold. It follows that theorems 2, 3 and 4 also hold for the α-product.

Definition 8. Let f and g be distributions and suppose that the α-product $f \overset{\alpha}{\circ} g$ exists and is equal to the distribution vector $\underline{h} = [h_0, h_1, \ldots, h_r, \ldots]$ on the open interval (a,b). We say that h_0 is the finite part of $f \overset{\alpha}{\circ} g$ and then write

$$\text{p.f. } f \overset{\alpha}{\circ} g = h_0$$

on the interval (a,b).

The following theorem holds, see [5].

Theorem 5. The 0-product $x_+^{\lambda} \overset{0}{\circ} x_-^{-\lambda-q}$ exists and

$$x_+^{\lambda} \overset{0}{\circ} x_-^{-\lambda-q} = \underline{h}(\lambda,q) = [h_0(\lambda,q), h_1(\lambda,q), \ldots, h_{q-1}(\lambda,q)]$$

for $q = 1,2, \ldots$ and $\lambda \neq 0, \pm 1, \pm 2, \ldots$, where

$$h_i(\lambda,q) = \begin{cases} -\dfrac{\pi \operatorname{cosec}(\pi\lambda)}{2(q-1)!}\,\delta^{(q-1)}, & i = 0, \\[2ex] \dfrac{\Gamma(\lambda+q-i)\pi\operatorname{cosec}(\pi\lambda)}{\Gamma(\lambda+q)(q-i-1)!}\,\rho_{i-1}\,\delta^{(q-i-1)}, & 1 \leq i \leq q-1, \end{cases}$$

Γ denotes the gamma function and

$$\rho_i = \rho^{(i)}(0)$$

for $i = 0,1,2, \ldots$.

In particular, the product $x_+^{\lambda} \circ x_-^{-\lambda-1}$ exists and

$$x_+^{\lambda} \circ x_-^{-\lambda-1} = -\tfrac{1}{2}\pi\operatorname{cosec}(\pi\lambda)\,\delta$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$.

We now prove the following theorem.

Theorem 6. Let λ, μ be real numbers such that $\lambda, \mu, \lambda+\mu \neq -1,-2, \ldots$ and $\lambda + \mu < -1$. Then the α-product $x_+^{\lambda} \overset{\alpha}{\circ} x_-^{\mu}$ exists and

$$(3)\, x_+^{\lambda} \overset{\alpha}{\circ} x_-^{\mu} = \underline{h}(\lambda,\mu) = [0, h_1(\lambda,\mu), \ldots, h_q(\lambda,\mu)],$$

where $q = [-\lambda - \mu]$, $\alpha = -\lambda - \mu - q - 1$,

$$h_i(\lambda,\mu) = \frac{B(\mu+1, \lambda+q-i+1)}{(q-i)!} (-1)^{q-i} a_{q-i}(\lambda,\mu)\,\delta^{(q-i)},$$

$$a_i(\lambda,\mu) = \frac{(-1)^p \Gamma(\lambda+\mu+i+2)}{\Gamma(\lambda+\mu+p+i+2)} \int_0^1 u^{\lambda+\mu+p+i+1} \rho^{(p)}(u)\,du$$

for $i = 1, \ldots, q$ and B denotes the beta function.

Proof. Suppose first of all that $\lambda > -1$ and choose positive integers p, q such that $-1 < \mu + p$ and $-1 < \lambda + \mu + q < 0$. Then

$$x_-^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+p+1)} (-1)^p (x_-^{\mu+p})^{(p)}$$

and putting

$$(x_-^{\mu})_n = x_-^{\mu} * \delta_n$$

we have

$$\frac{\Gamma(\mu+p+1)}{\Gamma(\mu+1)} (-1)^p (x_-^{\mu})_n = \begin{cases} \displaystyle\int_x^{1/n} (t-x)^{\mu+p} \delta_n^{(p)}(t)\,dt, & x \leq 1/n, \\ 0, & x > 1/n\,. \end{cases}$$

Thus

$$\frac{\Gamma(\mu+p+1)}{\Gamma(\mu+1)} (-1)^p \int_{-\infty}^{\infty} x_+^{\lambda} (x_-^{\mu})_n x^i\,dx =$$

$$= \int_0^{1/n} x^{\lambda+i} \int_x^{1/n} (t-x)^{\mu+p} \delta_n^{(p)}(t)\,dt\,dx$$

$$= \int_0^{1/n} \delta_n^{(p)}(t) \int_0^t x^{\lambda+i} (t-x)^{\mu+p}\,dx\,dt$$

$$= \int_0^{1/n} t^{\lambda+\mu+p+i+1} \delta_n^{(p)}(t) \int_0^1 v^{\lambda+i} (1-v)^{\mu+p}\,dv\,dt$$

$$= B(\lambda+i+1,\mu+p+1)\,n^{-\lambda-\mu-i-1}\int_0^1 u^{\lambda+\mu+p+i+1}\,\rho^{(p)}(u)\,du$$

$$= \frac{\Gamma(\lambda+i+1)\Gamma(\mu+p+1)}{\Gamma(\lambda+\mu+i+2)}\,n^{-\lambda-\mu-i-1}(-1)^p\,a_i(\lambda,\mu)$$

where the substitutions $x = tv$ and $nt = u$ have been made. It follows that

$$\int_{-\infty}^{\infty} x_+^{\lambda}(x_-^{\mu})_n x^i\,dx = B(\mu+1,\lambda+i+1)\,n^{-\lambda-\mu-i-1}a_i(\lambda,\mu).$$

Now let ϕ be an arbitrary test function with compact support. Then

$$(x_+^{\lambda},(x_-^{\mu})_n\phi) = \sum_{i=0}^{q-1}\int_{-\infty}^{\infty} x_+^{\lambda}(x_-^{\mu})_n\frac{x^i}{i!}\phi^{(i)}(0)\,dx + O(n^{-\lambda-\mu-q-1})$$

$$= \sum_{i=0}^{q-1}\frac{B(\mu+1,\lambda+i+1)}{i!}(-1)^i a_i(\lambda,\mu)(\delta^{(i)},\phi)\,n^{-\lambda-\mu-i-1} + O(n^{\alpha})$$

$$= \sum_{i=1}^{q}\frac{B(\mu+1,\lambda+q-i+1)}{(q-i)!}(-1)^{q-i}a_{q-i}(\lambda,\mu)(\delta^{(q-i)},\phi)n^{\alpha+i} + O(n^{\alpha})$$

and equation (3) follows for $\lambda > -1$, $q = 1,2,\ldots,\ \mu,\ \lambda+\mu \neq -1,-2,\ldots$ and $-1 < \lambda+\mu+q < 0$.

Now assume that equation (3) holds for $-p < \lambda < 1-p$, $q = 1,2,\ldots,\ \mu,\ \lambda+\mu \neq -1,-2,\ldots$ and $-1 < \lambda+\mu+q < 0$, where p is some positive integer. Then using theorem 2 we have

$$\lambda x_+^{\lambda-1} \overset{\alpha}{\circ} x_-^{\mu} = (x_+^{\lambda} \overset{\alpha}{\circ} x_-^{\mu})' + \mu x_+^{\lambda} \overset{\alpha}{\circ} x_-^{\mu-1}$$

$$= [0,h_1'(\lambda,\mu),\ \ldots\ ,h_q'(\lambda,\mu),0] +$$

$$+ \mu[0,h_1(\lambda,\mu-1),\ \ldots\ ,h_{q+1}(\lambda,\mu-1)].$$

Now for $i = 1,\ \ldots\ ,q$ we have

$$h_i'(\lambda,\mu) + \mu h_i(\lambda,\mu-1) = \frac{B(\mu+1,\lambda+q-i+1)}{(q-i)!}(-1)^{q-i}a_{q-i}(\lambda,\mu)\delta^{(q-i+1)} +$$

$$+ \mu \frac{B(\mu,\lambda+q-i+2)}{(q-i+1)!}(-1)^{q-i+1}a_{q-i+1}(\lambda,\mu-1)\delta^{(q-i+1)}$$

$$= \lambda \frac{B(\mu+1,\lambda+q-i+1)}{(q-i+1)!}(-1)^{q-i+1}a_{q-i+1}(\lambda-1,\mu)\delta^{(q-i+1)}$$

$$= \lambda h_i(\lambda-1,\mu).$$

Further

$$\begin{aligned}\mu h_{q+1}(\lambda,\mu-1) &= \mu B(\mu,\lambda+1)a_0(\lambda,\mu-1)\delta \\ &= \lambda B(\mu+1,\lambda)a_0(\lambda-1,\mu)\delta \\ &= \lambda h_{q+1}(\lambda-1,\mu)\end{aligned}$$

and equation (3) follows for $-p-1 < \lambda < -p$, $q = 1,2,\ldots$, μ, $\lambda+\mu \neq -1,-2,\ldots$ and $-1 < \lambda+\mu+q < 0$. Equation (3) now follows by induction for $\lambda < -1$, $q = 1,2,\ldots$, λ, μ, $\lambda+\mu \neq -1,-2,\ldots$ and $-1 < \lambda+\mu+q < 0$. This completes the proof of the theorem.

Corollary. Let λ, μ be real numbers such that λ, μ, $\lambda+\mu \neq -1,-2,\ldots$ and $\lambda+\mu < -1$. Then

$$\text{p.f. } x_+^{\lambda} \overset{\alpha}{\circ} x_-^{\mu} = 0.$$

The proof of this corollary is immediate.

Theorem 7. The α-product $x_+^{\lambda} \overset{\alpha}{\circ} x_+^{p}$ exists and

$$x_+^{\lambda} \overset{\alpha}{\circ} x_+^{p} = [x_+^{\lambda+p}, -(-1)^p h_1(\lambda,p), \ldots, -(-1)^p h_q(\lambda,p)] \tag{4}$$

for $p = 0,1,2,\ldots$ and $q = 1,2,\ldots$, where $\alpha = -\lambda-p-q-1$, $-1 < \alpha < 0$, $\lambda \neq -1,-2,\ldots$ and $h_i(\lambda,p)$ is as defined in theorem 4 for $i = 1,\ldots,q$.

Proof. Since x^p is an infinitely differentiable function for $p = 0,1,2, \ldots$ the product $x_+^\lambda \overset{\alpha}{\circ} x^p$ is defined in the normal sense, see [1], and

$$x_+^\lambda \overset{\alpha}{\circ} x^p = x_+^{\lambda+p} = x_+^\lambda \overset{\alpha}{\circ} (x_+^p + (-1)^p x_-^p).$$

It follows that

$$x_+^\lambda \overset{\alpha}{\circ} x_+^p = x_+^{\lambda+p} - (-1)^p x_+^\lambda \overset{\alpha}{\circ} x_-^p$$

$$= [x_+^{\lambda+p}, 0, \ldots, 0] - (-1)^p [0, h_1(\lambda,p), \ldots, h_q(\lambda,p)]$$

and equation (4) follows. This completes the proof of the theorem.

Corollary. Let λ be a real number such that $\lambda \neq -1,-2, \ldots$ and $\lambda + p < -1$, for $p = 0,1,2, \ldots$. Then

$$\text{p.f. } x_+^\lambda \overset{\alpha}{\circ} x_+^p = x_+^{\lambda+p}.$$

The proof of this corollary is also immediate.

We finally point out that the product $x_+^\lambda \circ x_-^\mu$ exists and $x_+^\lambda \circ x_-^\mu = 0$ for $\lambda + \mu > -1$, see theorem 6 of [4].

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REZIME

O α-PROIZVODU DISTRIBUCIJA

Koristeći račun neutrixa uveden je i analiziran opštiji proizvod distribucija nego u [4]. Specijalno, nadjen je "-proizvod" $x_+^\lambda \overset{\alpha}{\circ} x_-^\mu$ za $\lambda+\mu < -1$, $\lambda,\mu,\lambda+\mu \neq -1,-2,\ldots$ i $\alpha = -\lambda-\mu-1-[-\lambda-\mu]$.

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