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ZERO-CONTINUOUS FUNCTIONS ON K-, N-, AND COMPLETE SPACES

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ABSTRACT

Zero-continuous functions with the help of the family of all the summable sequences - Theorem 3.1. and with its subfamilies - Theorem 3.4. and Theorem 3.6. are characterized. These results made it possible to obtain the continuity of summable and supersummable continuous functions.

1. INTRODUCTION

In this paper we shall investigate zero-continuous functions (continuous at x = 0) defined on the convergence group and with the values in a convergence group, using the family of all the summable sequences and some of its subfamilies. We shall obtain as consequences some results from [4] and [8] on the comparison of K-, N- and complete convergences.

We shall introduce the notions of supersummable and summable continuous functions as a generalization of the corresponding notions from [2], [4] and [5]. We shall obtain some

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characterizations of such functions.

2. PRELIMINARY NOTIONS AND NOTATIONS

Let X be a non empty set. The convergence on X is a function $G: X^N \to 2^X$ or equivalently G is a subset of $X^N \times X$. If $x \in G((x_n))$, we shall write $x_n \to x(G)$.

We shall use in the paper the following conditions for a convergence ([1],[7]):

(S) For each $x \in X$, and $x_n = x(n \in N)$, $x_n \to x(G)$.

(F) If $x_n \to x(G)$, then $y_n \to x(G)$ for any subsequence (y_n) of (x_n).

(H) If $x_n \to x(G)$ and $x_n \to y(G)$, then $x = y$.

(U) If $x_n \not\to x(G)$, then there exists a subsequence of (x_n) such that $z_n \not\to x(G)$ for every subsequence (z_n) of (y_n).

Let X be an additive group.

(L) If $x_n \to x(G)$ and $y_n \to y(G)$, then $x_n - y_n \to x - y(G)$.

(K) If $x_n \to 0(G)$, then each subsequence (y_n) of (x_n) has a subsequence (z_n) such that $\sum_{k=1}^{n} z_k \to z(G)$ for some $z \in X$.

(N) If $x_n \to 0(G)$, then each subsequence of (x_n) has a subsequence (y_n) such that $\sum_{k=1}^{n} z_k \to z(G)$ for every subsequence (z_n) of (y_n) and some $z \in X$.

In papers [3] and [6] normed K- and N-spaces were constructed, respectively, which were not complete spaces.

Let $S(G)$ be the family of all summable sequences with respect to the convergence G on a group X, i.e. $(x_n) \in S(G)$ iff $\sum_{k=1}^{n} x_k \to x(G)$ for some $x \in X$.

Let $S_s(G)$ be the family of all subseries summable sequences with respect to the convergence G, i.e. $(x_n) \in S_s(G)$ means that each subsequence (x_{n_k}) of (x_n) is summable with respect to the convergence G.

3. ZERO-CONTINUOUS FUNCTIONS

Let X_1 and X_2 be two groups endowed with the convergences G_1 and G_2, respectively, such that they satisfy conditions S and L. A map f of X_1 into X_2 is called (sequentially) continuous at a point $x \in X_1$ if for any sequence (x_n) in X_1 such that $x_n \to x(G_1)$ holds $f(x_n) \to f(x)(G_2)$. A function $f: X_1 \to X_2$ is continuous at X_1 (or simply continuous), if for any $x \in X_1$ and for any sequence (x_n) in X_1 such that $x_n \to x(G_1)$ holds $f(x_n) \to f(x)(G_2)$.

We shall consider in this paper only those functions f which satisfy condition $f(0) = 0$.

Since the convergence G_1 satisfies conditions L and S, the preceding definition of a continuous function at a point $x \in X_1$ implies for the functions with the property $f(0) = 0$ that for any sequence (y_n) in X_1 such that $y_n \to 0(G_1)$ holds $f(y_n) \to 0(G_2)$. The last condition is equivalent to the continuity of f at $x = 0$ (zero-continuous function).

If a function $f: X_1 \to X_2$, $f(0) = 0$, is additive, then f is continuous, iff it is zero-continuous. But there exist also non-additive functions f, $f(0) = 0$, which are continuous iff they are zero-continuous. For example, a quasi-norm q on a group, i.e. functional $q: X \to R$ such that $q(0) = 0$, $q(-x) = q(x)$ and $q(x+y) \le q(x) + q(y)$. This follows from the inequality

$$|q(x+y) - q(x)| \le q(y).$$

THEOREM 3.1. *Let* f *be a map from an FLSK convergence group* (X_1,G_1) *into a LSU convergence group* (X_2,G_2) *such that* $f(0) = 0$. *The function* f *is zero-continuous iff* $f(x_n) \to 0(G_2)$ *for any* $(x_n) \in S(G_1)$.

PROOF. Suppose that the theorem is not true. Then there exists a sequence (x_n) from X_1 such that $x_n \to 0(G_1)$ and $f(x_n) \not\to 0(G_2)$. Since G_2 satisfies condition U, there exists a subsequence (y_n) of (x_n) such that for each subsequence (z_n) of (y_n) it holds that $f(z_n) \not\to 0(G_2)$. We have by the F property of convergence G_1 $y_n \to O(G_1)$. Since G_1 is a K-convergence, there exists a subsequence (v_n) of (y_n) such that $(v_n) \in S(G_1)$. By the assumption of theorem $f(v_n) \to O(G_2)$. A contradiction.

Taking specially the identity map on a group, we obtain as a consequence the following

COROLLARY 3.2. ([Theorem 1. in 8]). *Let G_1 and G_2 be two SL convergences on an additive group X such that G_1 satisfies the Urysohn property-U and G_2 is a K-convergence with F property. If $S(G_1) \supset S(G_2)$, then $G_1 \supset G_2$.*

COROLLARY 3.3. ([4, Proposition 3.3.28.a.d]). *Let u be a homomorphism of Hausdorff topological R-modules of G into H. If G is metrizable and complete, then the following assertions are equivalent:*

a) u *is continuous;*

d) $\lim_{n\to\infty} u(x_n) = 0$ *for any summable sequence* (x_n) *in* G.

If we make more restrictive assumptions on convergences, then we can characterize the zero-continuity with a subfamily of the family $S(G_1)$.

THEOREM 3.4. *Let f be a map from an FLSN convergence group (X_1,G_1) into an LSU convergence group (X_2,G_2), such that $f(0) = 0$. The function f is zero-continuous iff $f(x_n) \to O(G_2)$ for any $(x_n) \in S_s(G_1)$.*

The proof is analogous to the proof of Theorem 1.

COROLLARY 3.5. ([8, Theorem 2]). *Let G_1 and G_2 be two N-convergences on a group X which are topological. If $S_s(G_1) = S_s(G_2)$ then $G_1 = G_2$.*

Let X be a Hausdorff topological group which is metrizable. Let d be an invariant metric on X generating its topology. We denote by $S_a(G)$ the family of all absolutely summable sequences (x_n), i.e. such that

$$\sum_{n=1}^{\infty} d(x_n,0) < \infty$$

and (x_n) is summable.

THEOREM 3.6. *Let f be a map from a Hausdorff topological group* X_1 *into a Hausdorff topological group* X_2 *such that* $f(0) = 0$. *If* X_1 *is metrizable and complete, then function f is zero-continuous iff* $f(x_n) \to 0$ *as* $n \to \infty$ *for any* $(x_n) \in S_a(G_1)$.

PROOF. Suppose that the theorem is not true. Then there exists a sequence (x_n) from X_1 such that $d(x_n,0) \to 0$ as $n \to \infty$ and $q(f(x_n)) \not\to 0$ as $n \to \infty$ for some quasi-norm q which belongs to the family of quasi-norms which generates the topology of X_1. Then there exists a subsequence (y_n) of (x_n), such that for each subsequence (z_n) of (y_n) holds $q(f(z_n)) \not\to 0$ as $n \to \infty$. There exists a subsequence (u_n) of (y_n) such that $d(u_n,0) < 2^{-n}$ $(n \in N)$. Since X_1 is complete, the sequence (u_n) is summable and so $(u_n) \in S_a(G_1)$. Then by the assumption of the Theorem $q(f(u_n)) \to 0$ as $n \to \infty$, which is the expected contradiction.

COROLLARY 3.7. ([8, Theorem 3]). *Let* G_1 *and* G_2 *be two convergences generated by two Banach space norms, respectively. If* $S_a(G_1) \supset S_a(G_2)$, *then* $G_1 = G_2$.

4. SUMMABLE CONTINUOUS FUNCTIONS

DEFINITION 4.1. *A function* $f: X_1 \to X_2$ *is (super-) summable continuous if for each sequence* $(x_n) \in S(G_1)$ $((x_n) \in S_s(G_1))$, i.e. $\sum_{k=1}^{n} x_k \to x(G_1)$ *for some* $x \in X_1$, *hold*

$$(f(x_n)) \in S(G_2) \quad \textit{and} \quad \sum_{k=1}^{n} f(x_k) \to f(x)(G_2).$$

REMARK. Super-summable continuous maps were considered in [2, p. 819] for Hausdorff locally convex spaces under the name subcontinuous functions, in [5, Definition 8.6] on

Hausdorff commutative groups under the name Σ-continuous functions and in [4, Definition 3.3.18] as a member of the family $P^c(X_1,X_2)$ where X_1 and X_2 are Hausdorff topological R-modules.

PROPOSITION 4.2. *Any super-summable continuous function is additive. If* X_1 *and* X_2 *are two convergence real vector spaces, then any super-summable continuous function is linear.*

PROOF. The first part of the Proposition is obvious. The proof of the second part of the Proposition is analogous to the proof of Proposition 3.3.19. from [4].

It is obvious that any continuous additive function is summable continuous and that any summable continuous function is super-summable continuous.

THEOREM 4.3. *Let* X_1 *be a group endowed with an FLSK convergence* G_1 *and* X_2 *be a group endowed with an FLUS convergence* G_2*. Then a summable continuous function* f *from* X_1 *into* X_2 *is a continuous additive function.*

PROOF. We have by Definition 4.1. for any $(x_n)\in S(G_1)$

$$f(x_n) = \sum_{k=1}^{n} f(x_k) - \sum_{k=1}^{n-1} f(x_k) \to 0 \text{ as } n \to \infty.$$

Then the assertion of Theorem 4.3. follows by Theorem 1.

We obtain in the same way, now with Theorem 3.4.

THEOREM 4.4. *Let* X_1 *be a group endowed with an FLSN convergence and* X_2 *be a group endowed with an FLUS convergence* G_2*. Then a super-summable continuous function* f *from* X_1 *into* X_2 *is a continuous additive function.*

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REZIME

NULA NEPREKIDNE FUNKCIJE NAD K-, N-, I KOMPLETNIM PROSTORIMA

U radu se ispituju funkcije koje su neprekidne (sekvencijalno) u tački x = 0 a koje su definisane nad konvergentnim grupama i sa vrednostima u konvergentnoj grupi. Ove tzv. nula-neprekidne funkcije ispituju se pomoću familije svih sumabilnih nizova - Teorema 3.1. i sa njenim podfamilijama - Teoreme 3.4. i 3.6. Uvode se i ispituju i sumabilno i super-sumabilno neprekidne funkcije.

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