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A COMMON FIXED POINT THEOREM IN UNIFORMIZABLE SPACES

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ABSTRACT

In this paper a generalization of Theorem 1 from [1] in uniformizable spaces is proved.

Let us recall Theorem 1 from [1].

THEOREM A. *Let* (X,d) *be a complete metric space,* S *and* T *one to one continuous mappings from* X *into* X, A *a continuous mapping from* X *into* SX ∩ TX *and* A *commute with* S *and* T.

Suppose that the following conditions are satisfied:

1) *For every* $x \in X$ *there exists* $n(x) \in \mathbb{N}$ *so that for every* $y \in X$:

$$d(A^{n(x)}x, A^{n(x)}y) \leq \min\{q[d(Sx,Ty)]d(Sx,Ty), d(Tx,Sy)\}$$

where $q: [0,\infty) \to [0,1)$ *is a nondescreasing function such that*

$$\lim_{t\to\infty} t(1 - q(t)) = +\infty .$$

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2) For some $x_0 \in X$ *one of the sets:*

$$\{A^m T^p(x_0): p \in N, m \in \{0,1,\dots,n(x_0)-1\}$$

and

$$\{A^m S^p(x_0): p \in N, m \in \{0,1,\dots,n(x_0)-1\}$$

is bounded.
Then there exists one and only one element $y \in X$ *such that:*

$$y = Sy = Ty = Ay .$$

Further, we shall recall the definition of a uniformizable space.

Let X be an arbitrary set. A mapping $d: X \times X \to R^+ = [0,\infty)$ is called a pseudometric iff for every $x,y,z \in X$

1. $d(x,y) \geq 0$; $d(x,x) = 0$;
2. $d(x,y) = d(y,x)$;
3. $d(x,y) \leq d(x,z) + d(z,y)$.

A pair $(X,\{d_i\}_{i\in I})$, where d_i is a pseudometric for every $i \in I$, is called a uniformizable space.

The convergence of the sequences, the notions of a Cauchy sequence and completeness $(X,\{d_i\})_{i\in I}$ are introduced in the usual way [3].

Now, we shall prove a generalization of Theorem A in uniformizable spaces.

THEOREM. *Let* $(X,\{d_i\}_{i\in I})$ *be a complete Hausdorff uniformizable space,* $f:I \to I$, S *and* T *one to one continuous mappings from* X *into* X, $A: X \to SX \cap TX$ *be continuous so that* A *commutes with* S *and* T *and the following conditions are satisfied:*

1) For every $x \in X$ *there exists* $n(x) \in N$ *so that for every* $y \in X$ *and every* $i \in I$:

$$d_i(A^{n(x)}x, A^{n(x)}y) \leq$$

$$\leq \min q_i\{d_{f(i)}(Sx,Ty)[d_{f(i)}(Sx,Ty)]d_{f(i)}(Tx,Sy)\}$$

where $q_i: [0,\infty) \to [0,1)$ *is a nondecreasing function for every* $i \in I$ *so that for every* $t \in [0,\infty)$:

$$\sup_{k\in\mathbb{N}_0} q_{f^k(i)}(t) < 1 \qquad (\mathbb{N}_0 = \mathbb{N} \cup \{0\})$$

and

$$\lim_{t\to+\infty} [t - t \sup_{k\in\mathbb{N}_0} q_{f^k(i)}(t)] = +\infty.$$

2) There exists $x_0 \in X$ *so that for every* $i \in I$ *there is* $L^{(i)}$ *or* $\tilde{L}^{(i)}$ *so that*:

a) $$L^{(i)} \geq \max\left\{\sup_{k\in\mathbb{N}_0} d_{f^k(i)}(A^{n(x_0)}x_0, Sx_0),\ \sup_{\substack{\ell,k\in\mathbb{N}_0\\0\leq r<n(x_0)}} d_{f^k(i)}(A^rT^\ell x_0, Sx_0)\right\}$$

or

b) $$\tilde{L}^{(i)} \geq \max\left\{\sup_{k\in\mathbb{N}_0} d_{f^k(i)}(A^{n(x_0)}x_0, Tx_0),\ \sup_{\substack{\ell,k\in\mathbb{N}_0\\0\leq r<n(x_0)}} d_{f^k(i)}(A^rS^\ell x_0, Tx_0)\right\}.$$

Then there exists $y \in X$ *so that* $y = Ay = Sy = Ty$.

This common fixed point is unique with the property that the sequence

$$\{d_{f^n(i)}(y, Sx_0)\}\quad n \in \mathbb{N}$$

is bounded for every $i \in I$.

PROOF. Since $AX \subseteq SX \cap TX$ (as in [1]), we can define the sequence $\{x_n\}_{n\in\mathbb{N}_0}$, where x_0 is from the condition 2)a. of the Theorem, such that:

$$Sx_{2k} = A^{n(x_{2k-1})}x_{2k-1},\quad Tx_{2k-1} = A^{n(x_{2k-2})}x_{2k-2},$$

$k \in \mathbb{N}$. Let

$$y_m = \begin{cases} Tx_{2k-1} & m = 2k-1 \\ Sx_{2k} & m = 2k. \end{cases}$$

Further, for every $m \in \mathbb{N}_0$ and every $\ell \in \mathbb{N}$ we have:

$$d_{f^m(i)}(A^{n(x_{2k-1})}x_{2k-1}, A^\ell x_{2k}) =$$

$$= d_{f^m(i)}(A^{n(x_{2k-1})}x_{2k-1}, A^\ell S^{-1}Sx_{2k}) =$$

$$= d_{f^m(i)}(A^{n(x_{2k-1})}x_{2k-1}, A^{n(x_{2k-1})}S^{-1}A^\ell x_{2k-1}) \leq$$

$$\leq d_{f^{m+1}(i)}(Tx_{2k-1}, A^\ell x_{2k-1}) =$$

$$= d_{f^{m+1}(i)}(A^{n(x_{2k-2})}x_{2k-2}, A^\ell x_{2k-1}) =$$

$$= d_{f^{m+1}(i)}(A^{n(x_{2k-2})}x_{2k-2}, A^\ell T^{-1}Tx_{2k-1}) =$$

$$= d_{f^{m+1}(i)}(A^{n(x_{2k-2})}x_{2k-2}, A^{n(x_{2k-2})}T^{-1}A^\ell x_{2k-2}) \leq$$

$$\leq q_{f^{m+1}(i)}\left[d_{f^{m+2}(i)}(Sx_{2k-2}, A^\ell x_{2k-2})\right] \cdot$$

$$\cdot\ d_{f^{m+2}(i)}(Sx_{2k-2}, A^\ell x_{2k-2}) =$$

$$= q_{f^{m+1}(i)}\left[d_{f^{m+2}(i)}(A^{n(x_{2k-3})}x_{2k-3}, A^\ell x_{2k-2})\right] \cdot$$

$$\cdot\ d_{f^{m+2}(i)}(A^{n(x_{2k-3})}x_{2k-3}, A^\ell x_{2k-2}) \leq \dots \leq$$

$$(1)\qquad \leq \prod_{s=1}^{k} q_{f^{m+2s-1}(i)}\,[d_{f^{m+2k}(i)}(Sx_0, A^\ell x_0)] \cdot$$

$$\cdot\ d_{f^{m+2k}(i)}(Sx_0, A^\ell x_0)$$

Similarly:

$$d_{f^m(i)}(A^{n(x_{2k})}x_{2k}, A^\ell x_{2k+1}) \leq$$

$$(1')\qquad \leq \prod_{s=0}^{k} q_{f^{m+2s}(i)}\,[d_{f^{m+2k+1}(i)}(Sx_0, A^\ell x_0)] \cdot$$

$$\cdot\ d_{f^{m+2k+1}(i)}(Sx_0, A^\ell x_0)$$

so,

$$(2)\qquad d_i(y_{2k},y_{2k+1}) \le \prod_{s=1}^{k} q_{f^{2s-1}(i)}\left[d_{f^{2k}(i)}(Sx_0, A^{n(x_{2k})}x_0)\right]\cdot$$

$$\cdot\ d_{f^{2k}(i)}(Sx_0, A^{n(x_{2k})}x_0)$$

and

$$(2')\qquad d_i(y_{2k-1},y_{2k}) \le \prod_{s=0}^{k} q_{f^{2s}(i)}\left[d_{f^{2k-1}(i)}(Sx_0, A^{n(x_{2k-1})}x_0)\right]\cdot$$

$$\cdot\ d_{f^{2k-1}(i)}(Sx_0, A^{n(x_{2k-1})}x_0)\ .$$

Let us prove that the sequence

$$\{d_{f^k(i)}(Sx_0,\ A^mT^\ell x_0)\}\qquad m,k,\ell \in \mathbb{N}_0$$

is bounded.

Since x_0 is from condition 2.a, for every $i \in I$ there exists:

$$b_0^{(i)} = \max\Big\{\sup_{k\in\mathbb{N}_0} d_{f^k(i)}(A^{n(x_0)}x_0,\ Sx_0),$$

$$\sup_{\substack{\ell,k\in\mathbb{N}_0\\ 0\le r<n(x_0)}} d_{f^k(i)}(A^rT^\ell x_0,\ Sx_0)\Big\}\ .$$

Let $m = p\cdot n(x_0) + r$, where $0 \le r < n(x_0)$. We shall prove that for every $i \in I$ there exists a bounded sequence $\{b_p^{(i)}\}_{p\in\mathbb{N}_0}$ such that:

$$(3)\qquad d_{f^k(i)}(A^{p\cdot n(x_0)+r}\ T^\ell x_0, Sx_0) \le b_p^{(i)}$$

for every p,ℓ,k in $\mathbb{N}_0$ where:

$$b_p^{(i)} = \sup_{k\in\mathbb{N}_0} q_{f^k(i)}\left[b_{p-1}^{(i)}\right]\cdot b_{p-1}^{(i)} + b_0^{(i)},\quad p \in \mathbb{N}.$$

The proof of (3) will be given by induction in respect to $p \in \mathbb{N}$.

For $p = 1$ we have that:

$$d_{f^k(i)}(A^{n(x_0)+r}\, T^{\ell} x_0,\, Sx_0) \le$$

$$\le \quad q_{f^k(i)}[(d_{f^{k+1}(i)}(A^r\, T^{\ell+1} x_0,\, Sx_0)] \cdot$$

$$\cdot\, d_{f^{k+1}(i)}(A^r\, T^{\ell+1} x_0,\, Sx_0) + d_{f^k(i)}(A^{n(x_0)} x_0, Sx_0)$$

so

$$\sup_{k\in \mathbb{N}_0} d_{f^k(i)}(A^{n(x_0)+r} T^{\ell} x_0,\, Sx_0) \le$$

$$\le \sup_{k\in \mathbb{N}_0} q_{f^k(i)}[b_0^{(i)}]\, b_0^{(i)} + b_0^{(i)} = b_1^{(i)} .$$

Suppose that (3) is satisfied for some $p \in \mathbb{N}$ and for every $\ell \in \mathbb{N}_0, k \in \mathbb{N}_0$ and prove that:

$$d_{f^k(i)}(A^{(p+1)n(x_0)+r}\, T^{\ell} x_0,\, Sx_0) \le b_{p+1}^{(i)}$$

for every ℓ, $k \in \mathbb{N}_0$.

$$d_{f^k(i)}(A^{(p+1)n(x_0)+r}\, T^{\ell} x_0,\, Sx_0) \le$$

$$\le \quad q_{f^k(i)}[d_{f^k(i)}(A^{p\, n(x_0)+r}\, T^{\ell+1} x_0,\, Sx_0)] \cdot$$

$$\cdot \quad d_{f^{k+1}(i)}(A^{p\cdot n(x_0)+r}\, T^{\ell+1} x_0,\, Sx_0) +$$

$$+ \quad d_{f^k(i)}(A^{n(x_0)} x_0,\, Sx_0) \le$$

$$\le \quad q_{f^k(i)}\left[b_p^{(i)}\right]\cdot b_p^{(i)} + b_0^{(i)} \le$$

$$\le \sup_{k\in \mathbb{N}_0} q_{f^k(i)}\left[b_p^{(i)}\right]\cdot b_p^{(i)} + b_0^{(i)} = b_{p+1}^{(i)} .$$

Since all the functions q_i are nondecreasing, $\sup_{k\in \mathbb{N}_0} q_{f^k(i)}$ is nondecreasing too, so similarly as in [1] (using Lemma 2.2 [5]) we have that $\left\{b_p^{(i)}\right\}_{p\in\mathbb{N}}$ is bounded for every $i \in I$ and let

$$D_i = \sup_{\substack{k,\ell\in\mathbb{N}_0\\ m\in\mathbb{N}}} d_{f^k(i)}(A^m T^\ell x_0, Sx_0).$$

Now we have that:

$$d_i(y_{2k},y_{2k+1}) \le \prod_{s=1}^{k} q_{f^{2s-1}(i)}[D_i]D_i \le$$

$$\le \left(\sup_{j\in\mathbb{N}_0} q_{f^j(i)}[D_i]\right)^k D_i$$

and

$$d_i(y_{2k-1},y_{2k}) \le \prod_{s=0}^{k-1} q_{f^{2s}(i)}[D_i]\cdot D_i \le$$

$$\le \left(\sup_{j\in\mathbb{N}_0} q_{f^j(i)}[D_i]\right)^k D_i$$

which implies for $m = 2k$ (and for the other case as well) that:

$$d_i(y_m,y_{m+k}) \le \sum_{j=1}^{k} d_i(y_{m+j-1},y_{m+j}) \le$$

$$\le \left(\sup_{j\in\mathbb{N}_0} q_{f^j(i)}[D_i]\right)^k D_i + \left(\sup_{j\in\mathbb{N}_0} q_{f^j(i)}[D_i]\right)^{k+1} D_i + \ldots .$$

Since

$$\sup_{j\in\mathbb{N}_0} q_{f^j(i)}[D_i] \le q(i) < 1$$

$\{y_n\}$ is a Cauchy sequence and let $y = \lim_{n\to\infty} y_n$.

Further, $\{Sx_{2k}\}_{k\in\mathbb{N}}$ and $\{Tx_{2k-1}\}_{k\in\mathbb{N}}$ are subsequences of sequence $\{y_n\}_{n\in\mathbb{N}}$ so it follows that:

$$\lim_{k\to\infty} Sx_{2k} = \lim_{k\to\infty} Tx_{2k-1} = y .$$

Also, using the inequality (1) for $\ell = 1,2$ we obtain that:

$$d_i(Sx_{2k},Ax_{2k}) = d_i(A^{n(x_{2k-1})}x_{2k-1},Ax_{2k}) \le$$

$$\leq \prod_{s=1}^{k} q_{f^{2s-1}(i)}[d_{f^{2k}(i)}(Sx_0,Ax_0)]\cdot d_{f^{2k}(i)}(Sx_0,Ax_0) \leq$$

$$\leq \left(\sup_{j\in N_0} q_{f^j(i)}[D_i]\right)^k \cdot D_i$$

and

$$d_i(Sx_{2k},A^2x_{2k}) = d_i(A^{n(x_{2k-1})}x_{2k-1},A^2x_{2k}) \leq \ldots \leq$$

$$\leq \left(\sup_{j\in N_0} q_{f^j(i)}[D_i]\right)^k D_i$$

which implies that:

$$\lim_{k\to\infty} Ax_{2k} = \lim_{k\to\infty} A^2x_{2k} = \lim_{k\to\infty} Sx_{2k} = y .$$

From the continuity of A and S and the commutativity we obtain that:

$$Ay = A(\lim_{k\to\infty} Sx_{2k}) = S(\lim_{k\to\infty} Ax_{2k}) = Sy$$

and

$$y = \lim_{k\to\infty} A^2x_{2k} = A(\lim_{k\to\infty} Ax_{2k}) = Ay.$$

Similarly, it follows that $Ay = Ty$ and so we have that y is a common fixed point for mappings A, S and T.

Since:

$$d_{f^n(i)}(y_m,Sx_0) \leq \sum_{j=1}^{m-1} d_{f^n(i)}(y_j,y_{j+1}) + d_{f^n(i)}(y_1,Sx_0)$$

using inequalities (2) and (2') we have that:

$$d_{f^n(i)}(y,Sx_0) \leq M_i < +\infty$$

for every $n\in N_0$ so:

$$\sup_{n\in N_0} d_{f^n(i)}(y,Sx_0) \leq M_i \quad \text{for every } i \in I.$$

Now, suppose that there exists an another common fixed point z for A, S and T such that for any $i \in I$ and every $k \in \mathbb{N}_0$:

$$d_{f^k(i)}(z, Sx_0) \leq Q_i .$$

It follows that for every $k \in \mathbb{N}_0$:

$$d_{f^k(i)}(y,z) \leq T_i + Q_i$$

which implies that:

$$d_i(y,z) \leq q_i[d_{f(i)}(y,z)]\cdot d_{f(i)}(y,z) \leq$$

$$\leq q_i[T_i + Q_i]\cdot d_{f(i)}(y,z) \leq \ldots \leq$$

$$\leq \left(\sup_{n\in\mathbb{N}_0} q_{f^n(i)}[T_i + Q_i]\right)^n (T_i + Q_i)$$

for every $i \in I$. Since $\sup_{n\in\mathbb{N}_0} q_{f^n(i)}(T_i + Q_i) \leq q < 1$, we obtain that:

$$d_i(y,z) = 0 ,$$

for every $i \in I$ and so:

$$y = z.$$

Corollary. *Let* $(X,\{d_i\}_{i\in I})$ *be a complete Hausdorff uniformizable space,* $f : I \to I$, $A : X \to X$ *satisfying the following conditions:*

1. *For every* $i \in I$ *there exists* $q_i \geq 0$ *such that:*

$$d_i(Ax,Ay) \leq q_i d_{f(i)}(x,y) \textit{ for all } x,y \in X,$$

and for every $i \in I$:

$$\sup_{k\in\mathbb{N}_0} q_{f^k(i)} \leq q^{(i)} < 1.$$

2. *There exists* $x_0 \in X$ *such that for every* $i \in I$ *sequence*

$$\left\{d_{f^n(i)}(x_0, Ax_0)\right\}_{n \in N}$$

is bounded.

Then there exists one and only one solution of the equation $y = Ay$ *which also satisfies the condition:*

$$d_{f^n(i)}(y, x_0) \leq p(i) < +\infty \quad n \in \mathbb{N}.$$

Proof. For $S = T = Id$, $n(x) = 1$ for every $x \in X$ and $q_i(t) = q_i$ for every $t \in [0,\infty)$ all the conditions are satisfied so Theorem conclude our statement. If d_i, $i \in I$, are seminorms this corollary is, in fact, theorem 1 from [2].

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REZIME

TEOREMA O ZAJEDNIČKOJ NEPOKRETNOJ TAČKI U UNIFORMIZABILNIM PROSTORIMA

U ovom radu dokazano je uopštenje teoreme 1 iz [1] u uniformizabilnim prostorima. Slični rezultati su dobijeni u [2] i [3].

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