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PERTURBATIONS OF A LINEAR DIFFERENCE EQUATION

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ABSTRACT

The paper deals with an n-th order linear difference equation and gives conditions under which its linear perturbation preserves the same ℓ^p- affiliation.

1. INTRODUCTION

In the theory of ordinary differential equations there is a well known

Theorem (Weyl [1,2]). *Consider the equation*

(1) $$y'' + p(t)y = 0, \ t \geq 0,$$

along with the perturbed linear equation

(2) $$y'' + (p(t) + q(t))y = 0, \ t \geq 0,$$

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where both p(t) *and* q(t) *are locally integrable on* $[0,\infty)$.

Then, the following Weyl alternative holds: if all solutions of (1) are (not) in $L^2[0,\infty)$ *and* $q(t) \in L^2[0,\infty)$ *then all solutions of (2) are (not) in* $L^2[0,\infty)$.

Weyl's alternative result was extended by Patula and Wong [3], to include arbitrary $L^p[0,\infty)$-perturbation q(t). Also, Mironov [4] extended this theorem to the case of the delay-differential equation, while Wyrwinska [5] considered some more general classes of perturbations. Wong [6] obtained related results for a restrictive class of 2n-th order differential equations. All these results were generalized and improved in paper [7] to the general n-th order linear differential equation and corresponding linear and non-linear perturbations with or without retardations.

Recently, Butler and Rao [8, Theorem 4.1.] improved the result of Patula and Wong. Using a technique similar to this in paper [7], we are able to generalize and modify this result to a case of the general n-th order linear difference equation

(3) $$Dy \equiv y(n+m) + p_{n-1}(m)y(n+m-1) + \dots + p_0(m)y(m) = 0, \quad m = 0,1,\dots$$

and its linear perturbation of the form

(4) $$D_L y \equiv Dy + q_{n-1}(m)y(n+m-1) + \dots + q_0(m)y(m) = 0, \quad m = 0,1,\dots\ .$$

2. PRELIMINARIES

In this section we shall introduce some notation and list some well known results which will be used in the sequel.

Our first result is a discrete form of the variation of the parameters formula. The symbol Δ is a forward difference operator i.e. $\Delta y(m) = y(m+1) - y(m)$.

Lemma 1. *Suppose* $F(u,v)$ *is a continuous function on* R^2 *and* y *is the solution of* $Dy = F(m,y(m))$. *Then*

$$\Delta^j y(m) = \sum_{i=1}^{n} c_i \Delta^j y_i(m) + \sum_{s=0}^{m-1} \Delta_m^j G(s,m)F(s,y(s)),$$

$$j = 0,\dots,n-1,$$

where

$$G(s,m) = \frac{\begin{vmatrix} y_1(s+1 & \dots & y_n(s+1) \\ \cdot & \dots & \cdot \\ \cdot & \dots & \cdot \\ \cdot & \dots & \cdot \\ y_1(s+n-1) & \dots & y_n(s+n-1) \\ y_1(m) & & y_n(m) \end{vmatrix}}{\begin{vmatrix} y_1(s+1) & \dots & y_n(s+1) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ y_1(s+n) & & y_n(s+n) \end{vmatrix}} = \frac{W(s,m)}{D(s+1)} .$$

$D(s)$ is Casorati's determinant and for it we have

Lemma 2. (Heymann's theorem [9, p. 354-358]). *Casorati's determinant* $D(s)$ *satisfies the linear difference equation of the first order*

$$D(s+1) = (-1)^n p_0(s)D(s).$$

We have also need the following, obvious

Lemma 3. *If the sequence* $u \in \ell^p$ $(p \geq 1)$, *then it is also true for* Δu.

The next result is the discrete generalization of the fundamental inequality of Gronwall and Bellman as far as some non-linear variants of that inequality.

Lemma 4. (Willet and Wong [10]). *Suppose that* $v(n)$, $w(n)$ *and* $u(n+1)$, $(n = 0,1,\dots)$ *are non-negative sequences of numbers with* $v(0) = w(0) = 0$ *and that* u_0 *and* p *are constants such that* $u_0 > 0$ *and* $p \geq 0$, $p \neq 1$. *Then the inequality*

$$u(n+1) \leq u_0 + \sum_{j=0}^{n} v(j)u(j) + \sum_{j=0}^{n} w(j)u^p(j), \qquad n = 0,1,\dots$$

implies that

$$e(n)u(n+1) \leq \left[u_0^{1-p} + (1-p) \sum_{k=0}^{n} w(k)e^{1-p}(k)\right]^{\frac{1}{1-p}}, \qquad n = 0,1,\dots ,$$

where

$$e(n) = \prod_{j=0}^{n} (1 + v(j))^{-1}, \quad n = 0,1,\dots .$$

Lemma 5. (Hardy, Littlewood and Polya [11, p. 26]). *Let* $a_i \geq 0$ *for* $i = 1,\dots,m$. *Then*

$$\sum_{i=1}^{m} a_i^p \leq m^{1-p} \left(\sum_{i=1}^{m} a_i\right)^p \text{ for } 0 \leq p \leq 1,$$

$$m^{p-1} \sum_{i=1}^{m} a_i^p \geq \left(\sum_{i=1}^{m} a_i\right)^p \quad \text{for} \quad p \geq 1.$$

3. RESULTS

We are now able to give our main result.

Theorem 1. *Suppose that exists* $M > 0$ *such that*

$$\prod_{m=0}^{s} |p_0(m)| \geq M$$

and that all the solutions of (3) are in ℓ^p. *If* $q_i \in \ell^k$ *where*

$$1 \le k \le \infty \text{ for } 1 \le p \le 2$$

and

$$1 \le k \le \frac{p}{p-2} \text{ for } 2 < p \le \infty,$$

then all the solutions of (4) are in ℓ^p, *too.*

Proof. Starting with the representation given by Lemma 1 and using the fact that $G(s,m) = 0$ for $s = m-n+1,\ldots$ $\ldots,m-1$, we get

$$\Delta^j y(m) = \sum_{i=1}^{n} c_i \Delta^j y_i(m) + \sum_{i=1}^{n} \Delta^j y_i(m) \sum_{s=0}^{m-1} (-1)^{n+i} W_i(s) \frac{f(s)}{D(0) \prod_{k=0}^{s} p_0(k)}, \tag{5}$$

for $m = 0,1,\ldots$ and $j = 0,\ldots,m-1$.

Obviously,

$$|W_i(s)| \le C \sum_{\substack{j=1 \\ j\ne i}}^{n} |y_j(s+1)| \le C \sum_{j=1}^{n} |y_j(s+1)|,$$

for every $i = 1,\ldots,n$ and some $C > 0$. So, by (5) we get

$$|\Delta^j y(m)| \le C_1 \phi_j(m)[1 + \sum_{s=0}^{m-1} \phi_0(s+1)|f(s)|], \tag{6}$$

$$j = 0,\ldots,n-1; \quad m = 0,1,\ldots\ ,$$

where we denote $\phi_j(m) = \max\{|\Delta^j y_i(m)| \ :\ i = 1,\ldots,n\}$ for $j = 0,\ldots,n-1$.

As

$$f(s) = -\sum_{i=0}^{n-1} q_i(s) y(i+s) = \sum_{i=0}^{n-1} \bar{q}_i(s) \Delta^i y(s)$$

where $\bar{q}_i(s)$ are linear combinations of $q_i(a)$, relation (6) leads to

$$(7) \qquad |\Delta^j y(m)| \le C_1 \phi_j(m)[1 + \sum_{s=0}^{m-1} \phi_0(s+1)q(s) \sum_{i=0}^{n-1} |\Delta^i y(s)|],$$

$$j = 0,\dots,n-1; \quad m = 1,2,\dots,$$

where $q(s) = max\{|\bar{q}_i(s)|,\ i = 0,\dots,n-1\}$.

We shall put

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{k} + \frac{1}{k'} = 1$$

and differ the next cases:

a) $1 \le k \le p'$.

Relation (7) and the boundedness of the solutions of equation (3) imply

$$\sum_{j=0}^{n-1} |\Delta^j y(m)| \le C_2[1 + \sum_{s=0}^{m-1} \phi_0(s+1)q(s) \sum_{i=0}^{n-1} |\Delta^i y(s)|],$$

$$m = 1,2,\dots .$$

$\phi_0 \in \ell^p$ implies $\phi_0 \in \ell^{k'}$ and because of $q \in \ell^k$, $\phi\, q \in \ell$ such that

$$\sum_{j=0}^{n-1} |\Delta^j y(m)|$$

is bounded according to Lemma 4.

Using again relation (7), we obtain

$$|\Delta^j y(m)| \le C_1 \phi_j(m)[1 + C_3 \sum_{s=0}^{n-1} \phi_0(s+1)q(s)] \le$$

$$\le C_4 \phi_j(m), \quad j = 0,\dots,n-1; \quad m = 1,2,\dots$$

which implies that $\Delta^j y \in \ell^p$, $j = 0,\dots,n-1$.

b) $1 \le p \le 2 \le p' < k$

Using Hölder's inequality and Lemma 5, we get

$$\sum_{s=0}^{m-1} \phi_o(s+1)q(s) \sum_{i=0}^{n-1} |\Delta^j y(s)| \le$$

$$\le \left(\sum_{s=0}^{m-1} \phi_o(s+1)q(s))^{p'} \right)^{1/p'} \left(\sum_{s=0}^{m-1} \left(\sum_{i=0}^{n-1} |\Delta^i y(s)| \right)^p \right)^{\frac{1}{p}} \le$$

$$\le C_5 \left(\sum_{s=0}^{m-1} \phi_o^{p'}(s+1) \right)^{1/p'} \left(\sum_{s=0}^{m-1} \sum_{i=0}^{n-1} |\Delta^i y(s)|^p \right)^{1/p} \le$$

$$\le C_6 \left(\sum_{s=0}^{m-1} \phi_o^p(s+1) \right)^{1/p'} \left(\sum_{s=0}^{m-1} \sum_{i=0}^{n-1} |\Delta^i y(s)|^p \right)^{1/p} \le$$

$$\le C_7 \left(\sum_{s=0}^{m-1} \sum_{i=0}^{n-1} |\Delta^i y(s)|^p \right)^{1/p}.$$

Putting the last inequality in relation (7), we obtain

$$|\Delta^j y(m)| \le C_1 \phi_j(m) [1 + C_7 (\sum_{s=0}^{m-1} \sum_{i=0}^{n-1} |\Delta^i y(s)|^p)^{1/p}],$$

$$j = 0,\dots,n-1; \; m = 1,2,\dots ,$$

which, according to Lemma 5, implies

$$\text{(8)} \qquad \sum_{j=0}^{n-1} |\Delta^j y(m)|^p \le C_8 \sum_{j=0}^{n-1} \phi_j^p(m) [1 + \sum_{s=0}^{m-1} \sum_{i=0}^{n-1} |\Delta^i y(s)|^p],$$

$$m = 1,2,\dots .$$

Summing (8) from 0 to r−1 and using the fact that $\phi_j^p \in \ell$, Lemma 4 implies that all the solutions of (4) are in ℓ^p.

c) $1 < p' < 2 < p, \quad p' < k \le \frac{p}{p-2}$.

Now,

$$\sum_{s=0}^{m-1} (\phi_0(s+1)q(s))^{p'} \leq$$

$$\leq \Big(\sum_{s=0}^{m-1} \phi_0^p(s+1)\Big)^{1/p} \Big(\sum_{s=0}^{m-1} (q(s))^{\frac{pp'}{p-p'}}\Big)^{\frac{p-p'}{pp}} \leq$$

$$\leq C_9 \Big(\sum_{s=0}^{m-1} \phi_0^p(s+1)\Big)^{1/p} \Big(\sum_{s=0}^{m-1} q^k(s)\Big)^{\frac{p-p'}{pp}} \leq C_{10},$$

and the proof follows the same line as in case b).

If p,k satisfy the same conditions as in Theorem 1, we are able to give a converse result as follows.

Theorem 2. *Suppose that*

$$\prod_{m=0}^{s} |p_0(m)| \geq M > 0$$

and $q_i \in \ell^k$ *for* $i = 0,\dots,n-1$. *If some solution of (3) is not in* ℓ^p *then the same in true for some solution of (4).*

Proof. Suppose that all the solutions of (4) are in ℓ^p, which by Theorem 1 implies that all the solutions of (3) are in ℓ^p, since equation (3) can be considered as a linear perturbation of (4) i.e.

$$Dy = D_L y - \sum_{i=0}^{n-1} q_i(m)y(m+i).$$

The linear differential equation of the form (3) is of special interest when $p_{n-1}(t) \equiv 0$. In the theory of the difference equation, the adequate case is when $p_0(m) \equiv 1$. For that case, when operators D and D_L take, respectively, the form

(9) $$My \equiv y(n+m) + p_{n-1}(m)y(n+m-1) + \dots + p_1(m)y(m+1) + y(m) = 0,\ m = 0,1,\dots$$

and

$$(10) \qquad M_L y \equiv My + q_{n-1}(m)y(n+m-1) + \dots + q_1(m)y(m+1) = 0, \quad m = 0,1,\dots ,$$

we obtain from Theorem 1 and 2 the following

Corollary. *Suppose that* p,k *satisfy the same conditions as in Theorem 1. Then, all the solutions of equation (9) are in* ℓ^p *iff the same is true for all the solutions of (10).*

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REZIME

PERTURBACIJE LINEARNE DIFERENCNE JEDNAČINE

U radu se posmatra linearna diferencna jednačina oblika

(1) $$Dy = y(n+m) + p_{n-1}(m)y(n+m-1) + \ldots + p_0(m)y(m) = 0,$$

$$m = 0,1,\ldots$$

i njoj pridružena linearna perturbacija oblika

(2) $$D_L y = Dy + q_{n-1}(m)y(n+m-1) + \ldots + q_0(m)y(m) = 0,$$

$$m = 0,1,\ldots .$$

Ukoliko nizovi q_i pripadaju prostoru ℓ^k a sva rešenja jednačine (1) prostoru ℓ^p, uz odredjenu zavisnost parametara k i p, sva rešenja jednačine (2) takodje pripadaju ℓ^p.

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