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A NOTE ON SOME APPLICATIONS OF BELLMAN'S RESULTS ON ALMOST ORTHOGONAL SEQUENCES

Endre Pap

*University of Novi Sad, Faculty of Science,
Institute of Mathematics, Dr. I. Djuričića 4,
21000 Novi Sad, Yugoslavia*

ABSTRACT

In this paper an asymptotic version of the Riesz-Fisher theorem is proved. Convergence in the square mean is connected with the almost orthogonal sequences.

The following Riesz-Fisher theorem is well-known: Let $\{g_n\}$ be a sequence of orthogonal functions in $L_2(a,b)$, and $\{b_n\}$ be a sequence of numbers from ℓ_2, i.e.

$$\sum_{n=1}^{\infty} |b_n|^2 < \infty ,$$

then there exists a function $f \in L_2(a,b)$ such that

$$f = \sum_{n=1}^{\infty} b_n g_n \quad \text{in} \quad L_2(a,b)$$

and

$$b_n = \int_a^b f \, \bar{g}_n \, dx.$$

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We shall prove in this note the following asymptotic version of the Riesz-Fisher theorem.

Theorem 1. *Let* $\{F_n\}$ *be a sequence of functions from the space* $L_2(a,b)$ *(where* $[a,b]$ *is a finite real interval) which does not converge in the square mean to zero. If*

$$\lim_{j\to\infty} \int_a^b F_i(x)\,\overline{F_j(x)}\,dx = 0 \quad (i \in N),$$

then there exists a subsequence $\{g_k\}$ *of* $\{F_n\}$ *such that for any sequence* $\{b_k\}$ *from* ℓ_2*, there exists a function* $f \in L_2(a,b)$*, such that*

$$\sum_{k=1}^{\infty} \left|b_k - \frac{1}{\|g_k\|}\int_a^b f(x)\,\overline{g_k(x)}\,dx\right|^2 < \infty .$$

The main tool in the proof of the preceding Theorem 1 will be the following R. Bellman's theorem from [4].

Bellman theorem. *Let* $\{g_n\}$ *be on almost orthogonal sequence of functions on a finite interval* (a,b)*, i.e.*

$$\int_a^b |g_n|^2\,dx = 1$$

and

$$\sum_{m,n} |a_{mn}|^2 < \infty$$

where

$$a_{mn} = \begin{cases} \int_a^b g_n(x)\overline{g_m(x)}\,dx & n \neq m \\ 0 & n = m. \end{cases}$$

If a sequence $\{b_n\}$ *is from* ℓ_2*, then there exists a function* $f \in L_2(a,b)$ *such that*

$$\sum_{k=1}^{\infty} \left|b_k - \int_a^b f(x)\overline{g_k(x)}\,dx\right|^2 \leq \left(\sum_{n,m} |a_{nm}|^2\right)\left(\sum_{k=1}^{\infty} |b_k|^2\right).$$

We shall use in the proof of Theorem 1, also an elementary theorem from [1], which is of a similar nature as Theorem 1 from [2], see also the Basic Matrix Theorem 2 ([3], p. 7-10).

Diagonal theorem. *Let* $[x_{ij}]$ $(i,j \in \mathbb{N})$ *be a matrix of nonnegative real numbers such that*

$$\lim_{j\to\infty} x_{ij} = 0 \quad (i \in \mathbb{N}),$$

$$\lim_{i\to\infty} x_{ij} = 0 \quad (j \in \mathbb{N})$$

and

$$\lim_{i\to\infty} x_{ii} = 0.$$

Then there exists an infinite set I, $I \subset \mathbb{N}$, *such that*

$$\sum_{i,j\in I} x_{ij} < \infty.$$

Proof of Theorem 1. Let

$$h_n = \frac{F_n}{\| F_n \|},$$

where

$$\| F_n \| = \left(\int_a^b |F_n|^2 \, dx \right)^{\frac{1}{2}} \quad (n \in \mathbb{N}).$$

Since the sequence $\{F_n\}$ does not converge in the square mean to zero, there exist a number $m > 0$ and a subsequence $\{F_{n_k}\}$ such that $\| F_{n_k} \| \geq m > 0$ $(k \in \mathbb{N})$.

Hence

$$\left| \int_a^b h_{n_i} \bar{h}_{n_j} \, dx \right| \leq \frac{1}{m^2} \left| \int_a^b F_{n_i} \bar{F}_{n_j} \, dx \right|.$$

Let

$$x_{ij} = \int_a^b h_{n_i} \bar{h}_{n_j} \, dx \quad \text{for} \quad i \neq j \quad \text{and} \quad x_{ii} = 0.$$

Then, we obtain

$$|x_{ij}|^2 \to 0 \quad \text{for} \quad j \to \infty, \ (i \in N),$$

and

$$|x_{ij}|^2 \to 0 \quad \text{for} \quad i \to \infty \quad (j \in N).$$

So we can apply the diagonal theorem on $|x_{ij}|^2$ $(i,j \in N)$. Then there exists an increasing sequence $\{p_i\}$ of natural numbers such

$$\sum_{i,j} |x_{p_i p_j}|^2 < \infty .$$

Let $\{g_k\}$ be the subsequence of the sequence $\{F_n\}$ with the sequence of indexes $\{n_{p_k}\}$. Then we have

$$\sum_{\substack{i,j \\ i \neq j}} \frac{1}{\|g_i\|^2 \|g_j\|^2} \left| \int_a^b g_i \bar{g}_j \, dx \right|^2 < \infty .$$

This means, that $\{g_k / \|g_k\|\}$ is an almost orthogonal sequence. Then by Bellman's theorem we obtain the conclusion of the theorem.

For example, the sequence of functions $\{F_n\}$, where $F_n(x) = \exp(i\lambda_n x)$ and $\{\lambda_n\}$ is sequence of real numbers such that $|\lambda_n| \to \infty$, satisfies the preceding theorem.

Remark. Combining the Riesz-Fisher theorem, Bellman's theorem and Theorem 1., we have the following: Let $\{F_n\}$ be a sequence of functions from the space $L_2(a,b)$ such that

$$\int_a^b |F_n|^2 dx = 1 \quad \text{and} \quad \lim_{j \to \infty} \int_a^b F_i(x) \overline{F_j(x)} \, dx = 0 \ (i \in N).$$

Let $\{b_n\}$ be an arbitrary element from ℓ_2. Then there exists a function f from $L_2(a,b)$ such that either $F = \sum b_n F_n$ and

$$b_n = \int_a^b f\, \bar{F}_n\, dx \quad \text{or} \quad b_n \neq \int_a^b f\, \bar{F}_n\, dx$$

and either the series

$$\sum_{k=1}^{\infty} \left| b_k - \int_a^b f(x)\, \overline{F_k(x)}\, dx \right|^2$$

is convergent or divergent and in this second case, there exists a subsequence $\{g_k\}$ of $\{F_k\}$ for which the preceding series is finite.

Almost orthogonal sequences have another interesting property which is included in the next theorem.

Theorem 2. *Let* $\{g_n\}$ *be an almost orthogonal sequence on interval* $[a,b]$ *and* $f \in L_2(a,b)$. *If* $\{f_n\}$ *is a sequence from* $L_2(a,b)$ *which converges in the square mean to* f, *then*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \left(\int_a^b f_n(x)\overline{g_k(x)}dx \right)^2 = \sum_{k=1}^{\infty} \left(\int_a^b f(x)\overline{g_k(x)}dx \right)^2 .$$

The proof is based on the following Bellman's result from [4].

Generalized Bessel inequality

If $\{g_n\}$ is an almost orthogonal sequence on the interval $[a,b]$ and $f \in L_2(a,b)$, then we have

$$\sum_{k=1}^{\infty} \left| \int_a^b f(x)\overline{g_k(x)}dx \right|^2 \leq \left[1 + \left(\sum_{m,n} |a_{mn}|^2 \right)^{\frac{1}{2}}\right] \int_a^b |f(x)|^2 dx.$$

Proof of Theorem 2. We obtain by the generalized Bessel inequality

$$\sum_{k=1}^{\infty} \left| \int_a^b f_n(x)\overline{g_k(x)}dx \pm \int_a^b f(x)\overline{g_k(x)}dx \right|^2 \leq C \int_a^b |f_n \pm f|^2 dx,$$

where,

$$C = 1 + \left(\sum_{n,m} |a_{nm}|^2 \right)^{\frac{1}{2}}.$$

So we have

$$\left| \sum_{k=1}^{\infty} \left(\int_a^b f_n(x)\overline{g_k(x)}dx \right)^2 - \sum_{k=1}^{\infty} \left(\int_a^b f(x)\overline{g_k(x)}dx \right)^2 \right|^2 \leq$$

$$\leq \sum_{k=1}^{\infty} \left| \int_a^b f_n(x)\overline{g_k(x)}\, dx - \right.$$

$$\left. - \int_a^b f(x)\overline{g_k(x)}dx \right|^2 \cdot \sum_{k=1}^{\infty} \left| \int_a^b f_n(x)\overline{g_k(x)}dx + \right.$$

$$\left. + \int_a^b f(x)\overline{g_k(x)}dx \right|^2 \leq C^2 \int_a^b |f_n - f|^2\, dx \int_a^b |f_n + f|^2\, dx \leq$$

$$\leq C^2 \int_a^b |f_n - f|^2 dx \left\{ \left[\int_a^b |2f|^2 dx \right]^{\frac{1}{2}} + \left[\int_a^b |f_n - f|^2 dx \right]^{\frac{1}{2}} \right\} .$$

Letting $n \to \infty$, we obtain the assertion.

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REZIME

BELEŠKA O NEKIM PRIMENAMA BELLMANOVIH REZULTATA O SKORO ORTOGONALNIM NIZOVIMA

Na osnovu rezultata R. Bellmana [4] i Dijagonalne teoreme iz [1] dokazana je asimptotska verzija Riesz-Fisherove teoreme - Teorema 1.

U teoremi 2. je povezana konvergencija u srednjem kvadratnom sa skoro ortogonalnim nizovima.

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