ZBORNIK RADOVA
Prirodno-matematičkog fakulteta
Univerziteta u Novom Sadu
Serija za matematiku, 16,2(1986)

REVIEW OF RESEARCH
Faculty of Science
University of Novi Sad
Mathematics Series, 16,2(1986)

SOME RELATIONS FOR CURVATURE TENSORS IN A FINSLER SPACE

Irena Čomić
Faculty of Technical Sciences
21000 Novi Sad, Yugoslavia

ABSTRACT

This paper is a continuation of previous papers [5] and [6] by the same author. In [6] the induced connection coefficients which appear in (1.13), (1.14) and (1.15) are determined under conditions when $\bar{D}\ell^d$ and $\bar{D}\ell^k$ are defined by (1.18) and (1.19). In [6] it is proved that the mentioned formulae are consistent with each other only when relation (1.21) is satisfied. This condition is satisfied in the several cases. In this paper we shall examine the special case when $B^\alpha_a=B^\alpha_a(x)$ and $N^\alpha_k=N^\alpha_k(x)$ i.e. when B^α_a and N^α_k are not functions of $\dot{x}$. Since we suppose (1.1) i.e. that $g_{\alpha\beta}(x,\dot{x})\ B^\alpha_a(x)\ N^\alpha_k(x)=0$ so our examination is restricted only to those Finsler spaces in which the metric tensor has such a special form that relation (1.1) is valid. Let us denote such Finsler spaces by $\bar{F}_n$. The curvature tensors in $\bar{F}_n$ are defined by (2.5), (2.6) and (2.7). In this paper the relations between alternated differentials of a vector field and the curvature tensors are given. The curvature tensors and their alternated differentials are decomposed in the direction of vectors B^α_a and N^α_k.

AMS Mathematics Subject Classification (1980): Primary 53C60.
Key words and phrases: Finsler space, curvature tensors, absolute differential

PRELIMINARIES

In the Finsler space $\overline{F}_n$ the metric function is $L(x,\dot{x})$. Let us define m fields of vectors $B^\alpha_a(x)$ and n-m fields $N^\alpha_k(x)$ ($\alpha,\beta,\gamma,\delta,\varepsilon,\kappa,\ldots = 1,2,\ldots,n$; $a,b,c,d,e,f,i,j = 1,2,\ldots,m$; $k,l,m,n,p,q = m+1,\ldots,n$) in such a way that these vector fields are linearly independent at each x and satisfy the relations

(1.1) $$g_{\alpha\beta}B^\alpha_a N^\beta_k = 0 \quad \text{for each} \quad a = 1,2,\ldots,m, \quad k = m+1,\ldots,n.$$

Let us define

(1.2) $$g_{ab} = g_{\alpha\beta}B^\alpha_a B^\beta_b$$

(1.3) $$g_{k\ell} = g_{\alpha\beta}N^\alpha_k N^\beta_\ell$$

(1.4) $$B^b_\beta = g^{ab}\, g_{\alpha\beta}B^\alpha_a$$

(1.5) $$N^k_\beta = g^{km}\, g_{\alpha\beta}N^\alpha_m$$

$g_{\alpha\beta}$, B^a_α and N^k_α have zero degree of homogeneity in $\dot{x}$, (g^{ab}) and (g^{km}) are inverse matrices of (g_{ab}) and (g_{km}), respectively. From (1.3) and (1.5) we have

(1.6) $$N^k_\alpha N^\alpha_p = g^{k\ell}\, g_{\alpha\beta}N^\beta_\ell N^\alpha_p = g^{k\ell}g_{\ell p} = \delta^k_p.$$

Usually, we have that:

(1.7) $$\delta^\alpha_\beta = B^\alpha_a B^a_\beta + N^\alpha_k N^k_\beta .$$

The vectors dx and $\dot{x}$ are decomposed in the direction of vectors B^α_a and N^α_k in the following way

(1.8) $$dx^\alpha = B^\alpha_a\, du^a + N^\alpha_k\, dv^k$$

(1.9) $$\dot{x}^\alpha = B^\alpha_a\, \dot{u}^a + N^\alpha_k\, \dot{v}^k$$

We shall suppose that

$$0 = F^\alpha(x^1,\ldots,x^n,\ u^1,\ldots,u^m,\ v^{m+1},\ldots,v^n) \quad \alpha = 1,2,\ldots n,$$

any of the solutions of system of differential equation (1.8), together with (1.9) define x and $\dot{x}$ as the function of u, $\dot{u}$, v, $\dot{v}$ in the form

$$x^\alpha = x^\alpha \, (u^1,\dots, u^m, v^{m+1},\dots,v^n)$$

$$\dot{x}^\alpha = \dot{x}^\alpha \, (u^1,\dots,u^m, v^{m+1},\dots,v^n,\dot{u}^1,\dots\dot{u}^m,\dot{v}^{m+1},\dots,\dot{v}^n)$$

$$\alpha = 1,2,\dots,n$$

We shall suppose that the tensor and vector fields are homogeneous of degree zero in $\dot{x}$.For any vector field ξ, we have

$$\xi^\alpha = B^\alpha_a \, \xi^a + N^\alpha_k \, \xi^k. \tag{1.10}$$

Let us denote the absolute differential which corresponds to the motion from $(x,\dot{x})$ to $(x + dx, \dot{x} + d\dot{x})$ by D.Then,we have

$$D\xi^\alpha = (DB^\alpha_a)\xi^a + B^\alpha_a \, d\xi^a + (DN^\alpha_k)\xi^k + N^\alpha_k \, d\xi^k. \tag{1.11}$$

We shall use the notation

$$\ell^\alpha = L^{-1}(x,\dot{x}) \, \dot{x}^\alpha = L^{-1}(B^\alpha_a\dot{u}^a + N^\alpha_k \, \dot{v}^k) = B^\alpha_a\ell^a + N^\alpha_k\ell^k \tag{1.12}$$

where $\ell^a = L^{-1} \, \dot{u}^a$ and $\ell^k = L^{-1}\dot{v}^k$. From [6] we have

$$DB^\alpha_a = \bar{W}^d_a(d)B^\alpha_d + \bar{W}^k_a(d)N^\alpha_k, \tag{1.13}$$

$$DN^\alpha_m = \bar{W}^d_m(d)B^\alpha_d + \bar{W}^k_m(d)N^\alpha_k, \tag{1.14}$$

where

$$\bar{W}^y_x(d) = \bar{\Gamma}^*{}_x{}^y{}_b \, du^b + \bar{\Gamma}^*{}_x{}^y{}_n \, dv^n + \bar{A}_x{}^y{}_b \, \bar{D}\ell^b + \bar{A}_x{}^y{}_n \, \bar{D}\ell^n \tag{1.15}$$

$$x = a \ \text{or} \ x = m, \quad y = d \ \text{or} \ y = k.$$

The induced differentials $\bar{D}\xi^a$, $\bar{D}\xi^k$ are defined by $\bar{D}\xi^a = B^a_\alpha \, D\xi^\alpha$ $\bar{D}\xi^k = N^k_\alpha \, D\xi^\alpha$ and

$$D\xi^\alpha = B^\alpha_d \, \bar{D}\xi^d + N^\alpha_k \, \bar{D}\xi^k. \tag{1.16}$$

For ℓ^α we have

$$D\ell^\alpha = B^\alpha_d \, \bar{D}\ell^d + N^\alpha_k \, \bar{D}\ell^k, \tag{1.17}$$

where $\bar{D}\ell^d$ and $\bar{D}\ell^k$ will be defined by

(1.18) $\bar{D}\ell^d = d\ell^d + \bar{\Gamma}^{*d}_{o\ c}\, du^c + \bar{\Gamma}^{*d}_{o\ \ell} dv^\ell$

(1.19) $\bar{D}\ell^k = d\ell^k + \bar{\Gamma}^{*k}_{o\ c}\, du^c + \bar{\Gamma}^{*k}_{o\ \ell}\, dv^\ell$

(1.20) $\bar{\Gamma}^{*x}_{o\ y} = \bar{\Gamma}^{*x}_{a\ y}\ell^a + \bar{\Gamma}^{*x}_{m\ y}\ell^m = L^{-1}\bar{\Gamma}^{*x}_{y}$

$x = d$ or $x = k$, $\quad y = c$ or $y = \ell$.

As it was proved in [6], relation (1.17) is consistent with (1.18) - (1.20) iff

(1.21) $[(\dot{\partial}_\beta B^\alpha_a)\, \dot{u}^a + (\dot{\partial}_\beta\, N^\alpha_k)\, \dot{v}^k]\, D\ell^\beta = 0.$

In $\bar{F}_n$ this condition is obviously satisfied.

In [6] the induced connection coefficients $\bar{\Gamma}$ and $\bar{A}$ are determined. From $\ell_\alpha\, D\ell^\alpha = 0$, using (1.1), (1.17), we obtain

$g_{\alpha\beta}\ (B^\alpha_a\ell^a + N^\alpha_k\ell^k)(B^\beta_b\ \bar{D}\ell^b + N^\beta_\ell\ \bar{D}\ell^\ell) = g_{ab}\ell^a\ \bar{D}\ell^a + g_{k\ell}\ell^k\ \bar{D}\ell^k = 0$

i.e.

(1.22) $\ell_b\ \bar{D}\ell^b + \ell_\ell\,\bar{D}\ell^\ell = 0.$

For $\bar{D}\xi^a$ and $\bar{D}\xi^k$, we have

(1.23) $\bar{D}\xi^a = B^a_\alpha\ D\xi^\alpha = d\xi^a + \bar{W}^a_b(d)\ \xi^b + \bar{W}^a_k\ (d)\ \xi^k$

(1.24) $\bar{D}\xi^k = N^k_\alpha\ D\xi^\alpha = d\xi^k + \bar{W}^k_a(d)\,\xi^a + \bar{W}^k_m(d)\,\xi^m.$

From (1.16) it is obvious that $\bar{D}\xi^a$ and $\bar{D}\xi^k$ are components of $D\xi^\alpha$ in the directions of B^α_a and N^α_k, respectively.

2. ALTERNATED DIFFERENTIALS EXPRESSED BY CURVATURE TENSORS

If Δ is another absolute differential, corresponding to the motion from $(x,\dot{x})$ to $(x+\delta x, \dot{x}+\delta\dot{x})$, then we have

$$[\Delta D]\xi^{\alpha} = ([\Delta D]B^{\alpha}_{a})\xi^{a} + B^{\alpha}_{a}[\delta d]\xi^{a} + \tag{2.1}$$
$$([\Delta D]N^{\alpha}_{k})\xi^{k} + N^{\alpha}_{k}[\delta d]\xi^{k},$$

where

$$[\Delta D]B^{\alpha}_{a} = \bar{\Omega}_{a}{}^{e}(d\delta)B^{\alpha}_{e} + \bar{\Omega}^{m}_{a}(d\delta)N^{\alpha}_{m} + \mathcal{D}B^{\alpha}_{a} \tag{2.2}$$

$$[\Delta D]N^{\alpha}_{k} = \bar{\Omega}^{e}_{k}\ (d\delta)B^{\alpha}_{e} + \bar{\Omega}^{m}_{k}\ (d\delta)N^{\alpha}_{m} + \mathcal{D}N^{\alpha}_{k} \tag{2.3}$$

$$\bar{\Omega}_{x}{}^{y}(d,\delta) = \frac{1}{2}\bar{R}_{x}{}^{y}{}_{bc}[du^{b}\delta u^{c}] + \bar{R}_{x}{}^{y}{}_{bk}[du^{b}\delta v^{k}] + \frac{1}{2}\bar{R}_{x}{}^{y}{}_{k\ell}[dv^{k}\delta v] + \tag{2.4}$$
$$\bar{P}_{x}{}^{y}{}_{bc}[du^{b}\bar{\Delta}\ell^{c}] + \bar{P}_{x}{}^{y}{}_{bk}[du^{b}\bar{\Delta}\ell^{k}] + \bar{P}_{x}{}^{y}{}_{kc}[dv^{k}\bar{\Delta}\ell^{c}] + \bar{P}_{x}{}^{y}{}_{k\ell}\ [dv^{k}\bar{\Delta}\ell^{\ell}] +$$
$$\frac{1}{2}\bar{S}_{x}{}^{y}{}_{bc}[\bar{D}\ell^{b}\bar{\Delta}\ell^{c}] + \bar{S}_{x}{}^{y}{}_{bk}[\bar{D}\ell^{b}\bar{\Delta}\ell^{k}] + \frac{1}{2}\bar{S}_{x}{}^{y}{}_{k\ell}[\bar{D}\ell^{k}\bar{\Delta}\ell^{\ell}]$$

where $x \in \{a, k\}$ $\qquad y \in \{e, m\}$,

$$\bar{R}_{x}{}^{y}{}_{zw} = 2((\partial_{[w}\bar{\Gamma}^{*y}_{|x|\,z]} - \dot{\partial}_{d}\ \bar{\Gamma}^{*y}_{x[z}\bar{\Gamma}^{*d}_{w]} - \dot{\partial}_{\ell}\bar{\Gamma}^{*y}_{x[z}\ \bar{\Gamma}^{*\ell}_{w]} + \tag{2.5}$$
$$\bar{\Gamma}^{*d}_{x[z}\ \bar{\Gamma}^{*y}_{|d|w]} + \bar{\Gamma}^{*\ell}_{x[z}\ \bar{\Gamma}^{*y}_{|\ell|w]}) +$$
$$L^{-1}\ A_{x}{}^{y}{}_{d}\ (\partial_{[w}\bar{\Gamma}^{*d}_{z]} - \dot{\partial}_{f}\bar{\Gamma}^{*d}_{[z}\ \bar{\Gamma}^{*f}_{w]} - \dot{\partial}_{\ell}\ \bar{\Gamma}^{*d}_{[z}\ \bar{\Gamma}^{*\ell}_{w]}) +$$
$$L^{-1}\ A_{y}{}^{y}{}_{k}(\partial_{[w}\ \bar{\Gamma}^{*k}_{z]} - \dot{\partial}_{f}\ \bar{\Gamma}^{*k}_{[z}\ \bar{\Gamma}^{*f}_{w]} - \dot{\partial}_{\ell}\ \bar{\Gamma}^{*k}_{[z}\ \bar{\Gamma}^{*\ell}_{w]})),$$

$$\bar{P}_{x}{}^{y}{}_{zw} = L\dot{\partial}_{w}\ \bar{\Gamma}^{*y}_{xz} - \bar{A}_{x}{}^{y}{}_{w|z} + \bar{A}_{x}{}^{y}{}_{d}\ [(\dot{\partial}_{w}\ \bar{\Gamma}^{*d}_{bz})\ \dot{u}^{b} + (\dot{\partial}_{w}\ \bar{\Gamma}^{*d}_{n\ z})\ \dot{v}^{n}] + \tag{2.6}$$
$$\bar{A}_{x}{}^{y}{}_{n}\ [(\dot{\partial}_{w}\ \bar{\Gamma}^{*n}_{bz})\ \dot{u}^{b} + (\dot{\partial}_{w}\ \bar{\Gamma}^{*n}_{mz})\ \dot{v}^{m}],$$

$$\bar{S}_{x}{}^{y}{}_{zw} = 2(L\dot{\partial}_{[w}\ \bar{A}_{|x|z]}{}^{y} + \bar{A}_{x}{}^{d}{}_{[z}\ \bar{A}_{|d|}{}^{y}{}_{w]} + \bar{A}_{x}{}^{m}{}_{[z}\ \bar{A}_{|m|w]}{}^{y}) \tag{2.7}$$

and each of the indices x,y,z,w belong to one of the sets $\{a,b,c,d,e,\dots\}$ or $\{k,l,m,n,\dots\}$. $\mathcal{D}B^{\alpha}_{a}$ and $\mathcal{D}N^{\alpha}_{k}$ are infinitesimals of a higher order and have the form

(2.8) $\mathcal{D} B^{\alpha}_{a} = \bar{\theta}_{a}{}^{e}(d,\delta) B^{\alpha}_{e} + \bar{\theta}_{a}{}^{m}(d,\delta) N^{\alpha}_{m}$,

(2.9) $\mathcal{D} N^{\alpha}_{p} = \bar{\theta}_{p}{}^{e}(d,\delta) B^{\alpha}_{e} + \bar{\theta}_{p}{}^{m}(d,\delta) N^{\alpha}_{m}$,

(2.10)
$$\bar{\theta}_{y}{}^{x}(d,\delta) = [\bar{\Gamma}^{*}{}_{y}{}^{x}{}_{b} + L^{-1}(\bar{A}_{y}{}^{x}{}_{c}\, \bar{\Gamma}^{*}{}_{b}^{c} + \bar{A}_{y}{}^{x}{}_{k}\, \bar{\Gamma}^{*}{}_{b}^{k})](\delta d - d\delta) u^{b} +$$
$$[\bar{\Gamma}^{*}{}_{y}{}^{x}{}_{k} + L^{-1}(\bar{A}_{y}{}^{x}{}_{b}\, \bar{\Gamma}^{*}{}_{k}^{b} + \bar{A}_{y}{}^{x}{}_{n}\, \bar{\Gamma}^{*}{}_{k}^{n})](\delta d - d\delta)\, v^{k} +$$
$$\bar{A}_{y}{}^{x}{}_{b}(\delta d - d\delta)\ell^{b} + \bar{A}_{y}{}^{x}{}_{n}(\delta d - d\delta)\ell^{n},$$

$x \in \{e, m\}$, $y \in \{a, p\}$.

Introducing the notation

(2.11)
$$\bar{K}_{x}{}^{y}{}_{zw} = 2\,(\partial_{[w}\, \bar{\Gamma}^{*}{}_{|x|}{}^{y}{}_{z]} - \dot{\partial}_{d}\, \bar{\Gamma}^{*}{}_{x[z}^{y}\, \bar{\Gamma}^{*}{}_{w]}^{d} - \dot{\partial}_{\ell}\, \bar{\Gamma}^{*}{}_{x[z}^{y}\, \bar{\Gamma}^{*}{}_{w]}^{\ell} +$$
$$+ \bar{\Gamma}^{*}{}_{x[z}^{d}\, \bar{\Gamma}^{*}{}_{|d|w]}^{y} + \bar{\Gamma}^{*}{}_{x[z}^{\ell}\, \bar{\Gamma}^{*}{}_{\ell w]}^{y})$$

after some calculation, we obtain

(2.12)
$$L\bar{K}_{o}{}^{y}{}_{zw} \overset{\text{def}}{=} \bar{K}_{a}{}^{y}{}_{zw}\, u^{a} + \bar{K}_{k}{}^{y}{}_{zw}\, v^{k} = 2\,(\partial_{[w}\bar{\Gamma}^{*y}_{z]} -$$
$$- \dot{\partial}_{f}\, \bar{\Gamma}^{*y}_{[z}\, \Gamma^{*f}_{w]} - \dot{\partial}_{\ell}\, \bar{\Gamma}^{*y}_{[z}\, \bar{\Gamma}^{*\ell}_{w]}).$$

So (2.5) has the form

(2.13)
$$\bar{R}_{x}{}^{y}{}_{zw} = \bar{K}_{x}{}^{y}{}_{zw} + \bar{A}_{x}{}^{y}{}_{d}\, \bar{K}_{o}{}^{d}{}_{zw} + \bar{A}_{x}{}^{y}{}_{k}\, \bar{K}_{o}{}^{k}{}_{zw}$$

THEOREM 2.1. $[\bar{\Delta}\bar{D}]\xi^{a}$ and $[\bar{\Delta}\bar{D}]\xi^{k}$ are the components of $[\Delta D]\xi^{\alpha}$ in the direction B^{α}_{a} and N^{α}_{k} respectively. i.e.

(2.14)
$$[\Delta D]\xi^{\alpha} = B^{\alpha}_{a}\, [\bar{\Delta}\bar{D}]\xi^{a} + N^{\alpha}_{k}\, [\bar{\Delta}\bar{D}]\xi^{k}.$$

PROOF. Substituting (2.2), (2.3), (2.9) and (2.10) into (2.1), we obtain

$$[\Delta D]\xi^{\alpha} = [\bar{\Omega}_a{}^e(d,\delta)\xi^a + \bar{\Omega}_k{}^e(d,\delta)\xi^k]\, B_e^{\alpha} + [\bar{\Omega}_a{}^p(d,\delta)\xi^a + \bar{\Omega}_k{}^p(d,\delta)\xi^k]\, N_p^{\alpha} + \mathcal{D}\xi^{\alpha} \tag{2.15}$$

where

$$\mathcal{D}\xi^{\alpha} = ([\delta d]\xi^e + \bar{\Theta}_a{}^e(d,\delta)\xi^a + \bar{\Theta}_k{}^e(d,\delta)\xi^k)\, B_e^{\alpha} + ([\delta d]\xi^m + \bar{\Theta}_a{}^m(d,\delta)\xi^a + \bar{\Theta}_k{}^m(d,\delta)\xi^k)\, N_m^{\alpha}\,. \tag{2.16}$$

On the other hand, starting from (1.23) we have

$$[\bar{\Delta}\bar{D}]\xi^a = \delta(\bar{D}\xi^a) + \bar{w}_c{}^a(\delta)\bar{D}\xi^c + \bar{w}_k{}^a(\delta)\bar{D}\xi^k - d/\delta = (\delta\bar{w}_b{}^a(d) + \bar{w}_c{}^a(\delta)\bar{w}_b^c(d) + \bar{w}_n{}^a(\delta)\bar{w}_b{}^n(d) - d/\delta)\xi^b + (\delta\bar{w}_k{}^a(d) + \bar{w}_c{}^a(\delta)\bar{w}_k^c(d) + \bar{w}_n^a(\delta)\bar{w}_k{}^n(d) - d/\delta)\xi^k + [\delta d]\xi^a,$$

i.e.

$$[\bar{\Delta}\bar{D}]\xi^a = \bar{\Omega}_b{}^a(d,\delta)\xi^b + \bar{\Omega}_k{}^a(d,\delta)\xi^k + \mathcal{D}\xi^a\,, \tag{2.17}$$

where

$$\mathcal{D}\xi^a = [\delta d]\xi^a + \bar{\Theta}_b{}^a(d,\delta)\xi^b + \bar{\Theta}_k{}^a(d,\delta)\xi^k\,. \tag{2.18}$$

In a similar way we obtain

$$[\bar{\Delta}\bar{D}]\xi^k = \bar{\Omega}_b{}^k(d,\delta)\xi^b + \bar{\Omega}_m{}^k(d,\delta)\xi^m + \mathcal{D}\xi^k\,, \tag{2.19}$$

where

$$\mathcal{D}\xi^k = [\delta d]\xi^k + \bar{\Theta}_b{}^k(d,\delta)\xi^b + \bar{\Theta}_m{}^k(d,\delta)\xi^m\,. \tag{2.20}$$

Substituting (2.18) and (2.20) into (2.16), we get

$$\mathcal{D}\xi^{\alpha} = B_a^{\alpha}\mathcal{D}\xi^a + N_k^{\alpha}\mathcal{D}\xi^k. \tag{2.21}$$

Substituting (2.17), (2.19) and (2.21) into (2.15), we have (2.14) which proves Theorem 2.1.

THEOREM 2.2. $[\bar{\Delta}\bar{D}]\xi_a$ and $[\bar{\Delta}\bar{D}]\xi_k$ are the components of $[\Delta D]\xi_\alpha$ in the direction of B^a_α and N^k_α, respectively.

PROOF. For the covariant vector field we should have DB^a_α and DN^k_α. It may be proved that

$$(2.22)\qquad DB^a_\alpha = -\bar{W}_b{}^a(d)B_\alpha{}^b - \bar{W}_k{}^a(d)\,N^k_\alpha$$

$$(2.23)\qquad DN^k_\alpha = -\bar{W}_b{}^k(d)B^b_\alpha - \bar{W}_\ell{}^k(d)N_\alpha{}^\ell\ ,$$

and these formulae are consistent with (1.1) and (1.7).

If $\bar{D}\xi_a$ and $\bar{D}\xi_k$ are defined by

$$(2.24)\qquad D\xi_\alpha = B^a_\alpha\,\bar{D}\,\xi_a + N^k_\alpha\,\bar{D}\xi_k\ ,$$

then

$$(2.25)\qquad \bar{D}\xi_a = d\xi_a - \bar{W}_a{}^b(d)\xi_b - \bar{W}_a{}^k(d)\xi_k$$

$$(2.26)\qquad \bar{D}\xi_k = d\xi_k - \bar{W}_k{}^b(d)\xi_b - \bar{W}_k{}^\ell(d)\xi_\ell.$$

In a similar way as in Theorem 2.1, we obtain

$$(2.27)\qquad [\Delta D]\xi_\alpha = B^a_\alpha\ [\bar{\Delta}\bar{D}]\xi_a + N^k_\alpha\ [\bar{\Delta}\bar{D}]\xi_k,$$

where

$$(2.28)\qquad [\bar{\Delta}\bar{D}]\xi_a = -\bar{\Omega}_a{}^b(d,\delta)\xi_b - \bar{\Omega}_a{}^k(d,\delta)\xi_k\ ,$$

$$(2.29)\qquad [\bar{\Delta}\bar{D}]\xi_k = -\bar{\Omega}_k{}^b(d,\delta)\xi_b - \bar{\Omega}_k{}^\ell(d,\delta)\xi_\ell$$

3. ALTERNATED DIFFERENTIALS EXPRESSED BY COVARIANT DERIVATIONS

Starting from [5],

$$(3.1)\qquad \bar{D}\xi^a = \xi^a{}_{|b}\,du^b + \xi^a|_b\bar{D}\ell^b + \xi^a{}_{|m}dv^m + \xi^a|_m\,\bar{D}\ell^m,$$

$$(3.2)\qquad \bar{D}\xi^k = \xi^k{}_{|b}\,du^b + \xi^k|_b\,\bar{D}\ell^b + \xi^k{}_{|m}dv^m + \xi^k|_m\bar{D}\ell^m$$

and using

$$(3.3)\quad \begin{aligned} \bar{D}\xi^a &= B^a_\alpha D\xi^\alpha = B^a_\alpha (\xi^\alpha{}_{|\beta} dx^\beta + \xi^\alpha|_\beta D\ell^\beta), \\ \bar{D}\xi^k &= N^k_\alpha D\xi^\alpha = N^k_\alpha (\xi^\alpha{}_{|\beta} dx^\beta + \xi^\alpha|_\beta D\ell^\beta). \end{aligned}$$

Further, by (1.2),(1.3), we have

$$(3.4)\quad B^\beta_b \xi^\alpha{}_{|\beta} = B^\alpha_a \xi^a{}_{|b} + N^\alpha_m \xi^m{}_{|b},$$

$$(3.5)\quad N^\beta_k \xi^\alpha{}_{|\beta} = B^\alpha_a \xi^a{}_{|k} + N^\alpha_m \xi^m{}_{|k}$$

$$(3.6)\quad B^\beta_b \xi^\alpha|_\beta = B^\alpha_a \xi^a|_b + N^\alpha_m \xi^m|_b,$$

$$(3.8)\quad N^\beta_k \xi^\alpha|_\beta = B^\alpha_a \xi^a|_k + N^\alpha_m \xi^m|_k.$$

Using (1.7) and (3.4) - (3.7), we get

and (3.4)-(3.7), we get

$$(3.8)\quad \xi^\alpha|_\delta = B^a_\delta (B^\alpha_b \xi^b{}_{|a} + N^\alpha_k \xi^k{}_{|a}) + N^k_\delta (B^\alpha_a \xi^a{}_{|k} + N^\alpha_m \xi^m{}_{|k}).$$

$$(3.9)\quad \xi^\alpha|_\delta = B^a_\delta (B^\alpha_b \xi^b|_a + N^\alpha_k \xi^k|_a) + N^k_\delta (B^\alpha_b \xi^b|_k + N^\alpha_m \xi^m|_k).$$

From (3.3)-(3.6) or (3.8) and (3.9), we canñeasily obtain

$$(3.10)\quad \begin{aligned} \xi^a{}_{|b} &= \xi^\alpha{}_{|\beta} B^a_\alpha B^\beta_b \\ \xi^m{}_{|b} &= \xi^\alpha{}_{|\beta} N^m_\alpha B^\beta_b \\ &\vdots \\ \xi^m|_k &= \xi^\alpha|_\beta N^m_\alpha N^\beta_k . \end{aligned}$$

In the same way for the covariant vector field ξ_α, we have

$$\xi_{a|b} = \xi_{\alpha|\beta}\, B^{\alpha}_{a}\, B^{\beta}_{b}$$

(3.11) $$\xi_{m|b} = \xi_{\alpha|\beta}\, N^{\alpha}_{m}\, B^{\beta}_{b}$$

$$\vdots$$

$$\xi_{m}|_{k} = \xi_{\alpha}|_{\beta}\, N^{\alpha}_{m}\, N^{\beta}_{k}$$

We shall express $[\bar{\Delta}\bar{D}]\ \xi^{a}$ and $[\bar{\Delta}\bar{D}]\ \xi^{k}$ using the covariant derivations. Starting from (3.1), we have

$$\begin{aligned}\bar{\Delta}\bar{D}\xi^{a} = {} & \xi^{a}{}_{|b|c}\, du^{b}\delta u^{c} + \xi^{a}|_{b|c}\, \bar{D}\ell^{b}\delta u^{c} + \xi^{a}{}_{|m|c}\, dv^{m}\,\delta u^{c} + \xi^{a}|_{m|c}\, \bar{D}\ell^{m}\,\delta u^{c} + \\ & \xi^{a}{}_{|b|k}\, du^{b}\delta v^{k} + \xi^{a}|_{b|k}\, \bar{D}\ell^{b}\delta v^{k} + \xi^{a}{}_{|m|k}\, dv^{m}\delta v^{k} + \xi^{a}|_{m|k}\, \bar{D}\ell^{m}\delta v^{k} + \\ & \xi^{a}{}_{|b}|_{c}\, du^{b}\bar{\Delta}\ell^{c} + \xi^{a}|_{b}|_{c}\, \bar{D}\ell^{b}\bar{\Delta}\ell^{c} + \xi^{a}{}_{|m}|_{c}\, dv^{m}\bar{\Delta}\ell^{c} + \xi^{a}|_{m}|_{c}\, \bar{D}\ell^{m}\bar{\Delta}\ell^{c} + \\ & \xi^{a}{}_{|b}|_{k}\, du^{b}\, \bar{\Delta}\ell^{k} + \xi^{a}|_{b}|_{k}\, \bar{D}\ell^{b}\bar{\Delta}\ell^{k} + \xi^{a}{}_{|m}|_{k}\, dv^{m}\, \bar{\Delta}\ell^{k} + \xi^{a}|_{m}|_{k}\, \bar{D}\ell^{m}\bar{\Delta}\ell^{k} + \\ & \xi^{a}{}_{|n}\, \bar{\Delta}\bar{D}\, u^{b} + \xi^{a}|_{b}\, \bar{\Delta}\bar{D}\ell^{b} + \xi^{a}{}_{|m}\, \bar{\Delta}\bar{D}v^{m} + \xi^{a}|_{m}\, \bar{\Delta}\bar{D}\ell^{m}\,,\end{aligned}$$

from which we can get, after some calculation,

$$\begin{aligned}[\bar{\Delta}\bar{D}]\ \xi^{a} = {} & \xi^{a}_{[|b|c]}[du^{b}\ \delta u^{c}] + 2\ \xi^{a}_{[|b|_{c}]}\ [du^{b}\ \bar{\Delta}\ell^{c}] + \\ & 2\xi^{a}{}_{[|b|k]}[du^{b}\ \delta v^{k}] + 2\xi^{a}{}_{[|b|_{k}]}\ [du^{b}\ \bar{\Delta}\ell^{k}] + \\ & 2\xi^{a}{}_{[|k|_{b}]}\ [dv^{k}\ \bar{\Delta}\ell^{b}] + \xi^{a}{}_{[|k|\ell]}\ [dv^{k}\ \delta v^{\ell}] + \\ & 2\xi^{a}{}_{[|k|_{\ell}]}\ [dv^{k}\ \bar{\Delta}\ell^{\ell}] + \xi^{a}{}_{[|b|c]}\ [\bar{D}\ell^{b}\ \bar{\Delta}\ell^{c}] + \\ & 2\xi^{a}{}_{[|b|k]}\ [\bar{D}\ell^{b}\ \bar{\Delta}\ell^{k}] + \xi^{a}{}_{[|k|\ell]}\ [\bar{D}\ell^{k}\ \bar{\Delta}\ell^{\ell}] + B^{a},\end{aligned} \tag{3.12}$$

where

(3.13) $$B^{a} = \xi^{a}{}_{|d}[\bar{\Delta}\bar{D}]\ u^{d} + \xi^{a}|_{d}[\bar{\Delta}\bar{D}]\ \ell^{d} + \xi^{a}{}_{|k}\ [\bar{\Delta}\bar{D}]\ v^{k} + \xi^{a}|_{k}\ [\bar{\Delta}\bar{D}]\ \ell^{k}\ .$$

We shall first calculate B^{a}. As

$$\xi^a{}_{|d}[\bar{\Delta}\bar{D}]\ u^d = \xi^a{}_{|d}\ \Delta du^d - d/\delta =$$

$$\{\xi^a{}_{|d}\ [\delta du^d + (\bar{\Gamma}^{*d}_{b\ c}\ \delta u^c + \bar{\Gamma}^{*d}_{b\ k}\ \delta v^k + \bar{A}_b{}^d{}_c \bar{\Delta}\ell^c + \bar{A}_b{}^d{}_k \bar{\Delta}\ell^k).du^b$$

$$+ (\bar{\Gamma}^{*d}_{k\ c}\ \delta u^c + \bar{\Gamma}^{*d}_{k\ \ell}\delta v^\ell + \bar{A}_k{}^d{}_c\ \bar{\Delta}\ell^c + \bar{A}_k{}^d{}_\ell \bar{\Delta}\ell^\ell)dv^k]\} - \{d/\delta\}.$$

$$\xi^a{}_{|k}[\bar{\Delta}\bar{D}]v^k = \{\xi^a{}_{|k}[\delta dv^k + \bar{w}_b{}^k(\delta)du^b + \bar{w}\,{}^k_\ell(\delta)dv^\ell]\} - \{d/\delta\},$$

$$\xi^a|_b\ [\bar{\Delta}\bar{D}]\ell^b = \{\xi^a|_b[\delta\bar{D}\ell^b + \bar{w}_c{}^b(\delta)\bar{D}\ell^c + \bar{w}_k{}^b(\delta)\bar{D}\ell^k]\} - \{d/\delta\},$$

$$\xi^a|_m[\bar{\Delta}\bar{D}]\ell^m = \{\xi^a|_m\ [\delta\bar{D}\ell^m + \bar{w}_c{}^m(\delta)\bar{D}\ell^c + \bar{w}_k{}^m(\delta)\bar{D}\ell^k]\} - \{d/\delta\},$$

and using the evaluated expressions for $\delta\bar{D}\ell^b$, $\delta\bar{D}\ell^m$ from [5] we have

$$B^a = [\xi^a{}_{|d}\ \bar{\Gamma}^{*\ d}_{[b\ c]} + \xi^a{}_{|m}\ \bar{\Gamma}^{*\ m}_{[b\ c]} + \xi^a|_d \tfrac{1}{2}\bar{K}_o{}^d{}_{bc} + \tfrac{1}{2}\ \xi^a|_m \bar{K}_o{}^m{}_{b\ c}]\ [du^b\delta u^c] +$$

$$+ [2(\xi^a{}_{|d}\ \bar{\Gamma}^{*\ d}_{[b\ k]} + \xi^a|_m\ \bar{\Gamma}^{*\ m}_{[b\ k]}) + \xi^a|_d\ \bar{K}_o{}^d{}_{b\ k} + \xi^a|_m\ \bar{K}_o{}^m{}_{b\ k}][du^b\ \delta v^k] +$$

$$[\xi^a{}_{|d}\ \Gamma^{*\ d}_{[k\ell\ ]} + \xi^a{}_{|m}\ \Gamma^{*\ m}_{[k\ \ell]} + \tfrac{1}{2}\ \xi^a|_d\ \bar{K}_o{}^d{}_\ell + \tfrac{1}{2}\ \xi^a|_m\ \bar{K}_o{}^m{}_{k\ell}]\ \ [dv^k\delta v^\ell] +$$

$$[\xi^a{}_{|d}\ \bar{A}_b{}^d{}_c + \xi^a_{|m}\ \bar{A}_b{}^m{}_c - \xi^a|_d\ \bar{\Gamma}^{*\ d}_{\ c\,b} - \xi^a|_m\ \bar{\Gamma}^{*\ m}_{\ c\,b} + \xi^a|_d\ \dot{\partial}_c\ \bar{\Gamma}^{*\ d}_{\ b} +$$

$$+ \xi^a|_k\ \dot{\partial}_c\ \Gamma^{*\ k}_{\ b}]\cdot[du^b\ \bar{\Delta}\ell^c] +$$

(3.14) $$[\xi^a{}_{|d}\ \bar{A}_b{}^d{}_k + \xi^a{}_{|m}\ \bar{A}_b{}^m{}_k + \xi^a|_d(\dot{\partial}_k\bar{\Gamma}^{*d}_{\ b} - \bar{\Gamma}^{*\ d}_{k\ b}) + \xi^a|_m\ (\dot{\partial}_k\ \bar{\Gamma}^{*m}_{\ b} -$$

$$- \bar{\Gamma}^{*\ m}_{k\ b})]\ [du^b\ \bar{\Delta}\ell^k] +$$

$$[\xi^a{}_{|d}\ \bar{A}_k{}^d{}_b + \xi^a{}_{|m}\ \bar{A}_k{}^m{}_b + \xi^a|_d(\dot{\partial}_b\ \bar{\Gamma}^{*d}_k - \bar{\Gamma}^{*\ d}_{b\ k}) + \xi^a|_m\ (\dot{\partial}_b\bar{\Gamma}^{*m}_k -$$

$$- \bar{\Gamma}^{*m}_{b\ k})]\ [dv^k\bar{\Delta}\ell^b] +$$

$$[\xi^a{}_{|d}\ \bar{A}_k{}^d{}_\ell + \xi^a{}_{|m}\ \bar{A}_k{}^m{}_\ell + \xi^a|_d\ (\dot{\partial}_k\ \Gamma^{*d}_{\ \ell} - \bar{\Gamma}^{*\ d}_{k\ \ell}) + \xi^a|_m(\dot{\partial}_k\Gamma^{*m}_{\ \ell} - \bar{\Gamma}^{*\ m}_{k\ \ell})]\ [dv^k\bar{\Delta}\ell^\ell]$$

$$[\xi^a|_d \bar{A}_{[b}{}^d{}_{c]} + \xi^a|_m \bar{A}_{[b}{}^m{}_{c]}][\bar{D}\ell^b \bar{\Delta}\ell^c] + [\xi^a|_d \bar{A}_b{}^d{}_k + \xi^a|_m \bar{A}_b{}^m{}_k][\bar{D}\ell^b \bar{\Delta}\ell^k] +$$

$$[\xi^a|_d \bar{A}_{[k}{}^d{}_{\ell]} + \xi^a|_m \bar{A}_{[k}{}^m{}_{\ell]}][\bar{D}\ell^k \bar{\Delta}\ell^\ell] + \mathcal{D}_1 \xi^a ,$$

where

(3.15)
$$\mathcal{D}_1 \xi^a = [\xi^a{}_{|b} + \xi^a|_d L^{-1}\bar{\Gamma}^{*d}_b + \xi^a|_m L^{-1} \bar{\Gamma}^{*m}_b][\delta d]u^b +$$
$$[\xi^a{}_{|k} + \xi^a|_d L^{-1} \bar{\Gamma}^{*d}_k + \xi^a|_m L^{-1} \bar{\Gamma}^{*m}_k][\delta d]v^k +$$
$$\xi^a|_b [\delta d]\ell^b + \xi^a|_k [\delta d]\ell^k .$$

Writing the coefficient explicitly beside $[\delta d]u^b$ in (3.15) we get the expression

$$\partial_b \xi^a - \dot{\partial}_d \xi^a \bar{\Gamma}^{*d}_b - \dot{\partial}_k \xi^a \bar{\Gamma}^{*k}_b + \bar{\Gamma}^{*a}_{c\ b} \xi^c + \bar{\Gamma}^{*a}_{k\ b} \xi^k + L^{-1} (\dot{\partial}_d \xi^a + \bar{A}_c{}^a{}_d \xi^c + \bar{A}_k{}^a{}_d \xi^k) \bar{\Gamma}^{*d}_b +$$

$$+ L^{-1}(\dot{\partial}_k \xi^a + \bar{A}_c{}^a{}_k \xi^c + \bar{A}_\ell{}^a{}_k \xi^\ell)\Gamma^{*k}_b ,$$

which is the same as the coefficient beside $[\delta d]\ u^b$ in (2.17) because we have

$$[\delta d]\xi^a = \partial_b \xi^a [\delta d] u^b + \partial_k \xi^a [\delta d] v^k + L\dot{\partial}_b \xi^a [\delta d]\ell^b + L\dot{\partial}_n \xi^a [\delta d]\ell^n +$$
$$(\ell^b \dot{\partial}_b \xi^a + \ell^n \dot{\partial}_n \xi^a) [\delta d]L$$

under condition that ξ^a is homogeneous of degree zero in u^b and v^k i.e.

(3.16) $$\ell^b \dot{\partial}_b \xi^a + \ell^n \dot{\partial}_n \xi^a = 0.$$

Comparing the coefficients of $[\delta d]v^k$, $[\bar{\Delta}\bar{D}]\ell^b$ and $[\bar{\Delta}\bar{D}]\ell^n$ in (3.15) and (2.17), we get

(3.17) $$\mathcal{D}\xi^a = \mathcal{D}_1 \xi^a$$

From (3.12)(3.14) and (2.16), we get

$$(3.18)\qquad \xi^{a}_{[|x|y]} + \xi^{a}_{|d}\, \bar{\Gamma}^{*\,d}_{[x\,y]} + \xi^{a}_{|m}\bar{\Gamma}^{*\,m}_{[x\,y]} = \frac{1}{2}\,(\bar{R}_{d}{}^{a}{}_{xy}\,\xi^{d} + \bar{R}_{m}{}^{a}{}_{xy}\,\xi^{m} + \xi^{a}|_{d}\,\bar{K}_{o}{}^{d}{}_{xy} + \xi^{a}|_{m}\,\bar{K}_{o}{}^{m}{}_{xy}),$$

$$(3.19)\qquad 2\xi^{a}_{[|x|y]} + \xi^{a}_{|d}\,\bar{A}_{x}{}^{d}{}_{y} + \xi^{a}_{|m}\,\bar{A}_{x}{}^{m}{}_{y} + \xi^{a}|_{d}\,(\dot{\partial}_{y}\,\bar{\Gamma}^{*d}_{x} - \bar{\Gamma}^{*}{}_{y}{}^{d}{}_{x}) + \xi^{a}|_{m}\,(\dot{\partial}_{y}\,\bar{\Gamma}^{*m}_{x} - \bar{\Gamma}^{*}_{yx}) = \bar{P}_{d}{}^{a}{}_{xy}\,\xi^{d} + \bar{P}_{m}{}^{a}{}_{xy}\,\xi^{m},$$

$$(3.20)\qquad \xi^{a}{}_{[}|_{x}|_{y]} + \xi^{a}|_{d}\,\bar{A}_{[x}{}^{d}{}_{y]} + \xi^{a}|_{m}\,\bar{A}_{[x}{}^{m}{}_{y]} = \frac{1}{2}\,\bar{\bar{S}}_{a}{}^{d}{}_{xy}\,\xi^{a} + \frac{1}{2}\,S_{m}{}^{a}{}_{xy}\,\xi^{m},$$

$$x \in \{b,\, k\} \qquad y \in \{c,\, \ell\}$$

Relations (3.18)-(3.20) are valid if index a is substituted in them by $p(p = m+1,\ldots n)$.

4. DOUBLE ALTERNATED DIFFERENTIALS OF CURVATURE TENSORS

In [5] we obtained the relations which connect the curvature tensors of the Finsler space F_n and its subspaces F_m and F_{n-m}. These formulae have the form

$$\bar{R}_{a\,d\,b\,c} = R_{\delta\kappa\beta\gamma}B^{\delta\,\kappa\,\beta\,\gamma}_{a\,d\,b\,c}$$

$$\bar{R}_{a\,d\,b\,k} = R_{\delta\kappa\beta\gamma}B^{\delta\,\kappa\,\beta\,\gamma}_{a\,d\,b\,c}$$

$$\vdots$$

$$R_{k\,\ell\,m\,n} = R_{\delta\kappa\beta\gamma}N^{\delta\,\kappa\,\beta\,\gamma}_{k\,\ell\,m\,n}$$

and valid for the tensors P and S. Using the relation

$$(4.2)\qquad \delta^{\alpha}_{\beta} = B^{\alpha}_{a}\,B^{a}_{\beta} + N^{\alpha}_{k}\,N^{k}_{\beta}$$

several times, we obtain

$$R_{\alpha\beta\gamma\delta} = \bar{R}_{abcd} B^{abcd}_{\alpha\beta\gamma\delta} + \bar{R}_{abcn} B^{abc}_{\alpha\beta\gamma} N^{n}_{\delta} + \bar{R}_{abmd} B^{ab}_{\alpha\beta} N^{m}_{\gamma} B^{d}_{\delta} +$$

$$\bar{R}_{a\ell cd} B^{a}_{\alpha} N^{\ell}_{\beta} B^{cd}_{\gamma\delta} + \bar{R}_{kbcd} N^{k}_{\alpha} B^{bcd}_{\beta\gamma\delta} + \bar{R}_{abmn} B^{ab}_{\alpha\beta} N^{mn}_{\gamma\delta} +$$

$$\bar{R}_{a\ell cn} B^{a}_{\alpha} N^{\ell}_{\beta} B^{c}_{\gamma} N^{n}_{\delta} + \bar{R}_{kbcn} N^{k}_{\alpha} B^{bc}_{\beta\gamma} N^{n}_{\delta} + \bar{R}_{a\ell md} B^{a}_{\alpha} N^{\ell m}_{\beta\gamma} B^{d}_{\delta} +$$

(4.3)

$$\bar{R}_{kbnd} N^{k}_{\alpha} B^{b}_{\beta} N^{n}_{\gamma} B^{d}_{\delta} + \bar{R}_{k\ell cd} N^{k\ell}_{\alpha\beta} B^{cd}_{\gamma\delta} + \bar{R}_{a\ell mn} B^{a}_{\alpha} N^{\ell mn}_{\beta\gamma\delta} +$$

$$\bar{R}_{kbmn} N^{k}_{\alpha} B^{b}_{\beta} N^{mn}_{\gamma\delta} + \bar{R}_{k\ell cn} N^{k\ell}_{\alpha\beta} B^{c}_{\gamma} N^{n}_{\delta} + \bar{R}_{k\ell md} N^{k\ell m}_{\alpha\beta\gamma} B^{d}_{\delta} +$$

$$R_{k\ell mn} N^{k\ell mn}_{\alpha\beta\gamma\delta}$$

Relation (4.3) is valid for the tensors P and S also.

If the absolute differentials D_1 and D_2 are defined in a similar fashion as D and Δ, we obtain from (4.3).

(4.4) $$[D_1D_2]\, R_{\alpha\beta\gamma\delta} = \bar{R}_{abcd} ([D_1D_2] B^{a}_{\alpha}) B^{bcd}_{\beta\gamma\delta} +$$

$$\bar{R}_{abcd} B^{a}_{\alpha} ([D_1D_2] B^{b}_{\beta}) B^{cd}_{\gamma\delta} + \dots\dots\dots\dots +$$

$$\bar{R}_{kbcd} ([D_1D_2] N^{k}_{\alpha}) B^{bcd}_{\beta\gamma\delta} + \dots\dots + \bar{R}_{k\ell mn} N^{k1m}_{\alpha\beta\gamma} [D_1D_2] N^{n}_{\delta}$$

There are 16.4 = 64 summands of the right hand side of (4.4). Using relation (4.1), we have

(4.5)
$$B^{\alpha}_{a} [D_1D_2]\, B^{a}_{\beta} + N^{\alpha}_{k} [D_1D_2]\, N^{k}_{\beta} =$$
$$-([D_1D_2]\, B^{\alpha}_{a})\, B^{a}_{\alpha} - ([D_1D_2] N^{\alpha}_{k})\, N^{k}_{\beta}$$

Using (4.5), (2.2), (2.3) and summing the terms which appear in (4.4), we obtain

$$\bar{R}_{a\,b\,c\,d}\,([D_1D_2]B^a_\alpha)B^b_\beta{}^c_\gamma{}^d_\delta + \bar{R}_{k\,b\,c\,d}\,([D_1D_2]N^k_\alpha)\,B^b_\beta{}^c_\gamma{}^d_\delta =$$

$$\bar{R}_{\varepsilon\,b\,c\,d}\,B^b_\beta{}^c_\gamma{}^d_\delta\,(B^\varepsilon_a\,[D_1D_2]\,B^a_\alpha + N^\varepsilon_k\,[D_1D_2]\,N^k_\alpha) =$$

$$\bar{R}_{\varepsilon\,b\,c\,d}\,B^b_\beta{}^c_\gamma{}^d_\delta\,(-([D_1D_2]B^\varepsilon_a)\,B^a_\alpha - ([D_1D_2]\,N^\varepsilon_k)\,N^k_\alpha) =$$

$$\bar{R}_{\varepsilon\,b\,c\,d}\,B^b_\beta{}^c_\gamma{}^d_\delta\,(-\bar{\Omega}_a{}^e\,(d_2d_1)\,B^\varepsilon_e - \bar{\Omega}_a{}^p(d_2d_1)\,N^\varepsilon_p)\,B^a_\alpha +$$

$$\bar{R}_{\varepsilon\,b\,c\,d}\,B^b_\beta{}^c_\gamma{}^d_\delta\,(-\bar{\Omega}_k{}^e(d_2d_1)\,B^\varepsilon_e - \bar{\Omega}_k{}^p\,(d_2d_1)\,N^\varepsilon_p)\,N^k_\alpha =$$

$$-\bar{R}_{e\,b\,c\,d}\,\bar{\Omega}_a{}^e\,(d_2d_1)\,B^a_\alpha{}^b_\beta{}^c_\gamma{}^d_\delta - \bar{R}_{p\,b\,c\,d}\,\bar{\Omega}_a{}^p(d_2d_1)\,B^a_\alpha{}^b_\beta{}^c_\gamma{}^d_\delta -$$

$$-\bar{R}_{e\,b\,c\,d}\,\bar{\Omega}_k{}^e\,(d_2d_1)\,N^k_\alpha\,B^b_\beta{}^c_\gamma{}^d_\delta - \bar{R}_{p\,b\,c\,d}\,\bar{\Omega}_k{}^p\,(d_2d_1)\,N^k_\alpha\,B^b_\beta{}^c_\gamma{}^d_\delta .$$

There are 16.2 pair of summands on the right hand side of (4.4) which give by using the above method 16.2.4 summands. Since

$$[\bar{D}_1\bar{D}_2]\,\bar{R}_{x\,y\,u\,w} = -\bar{R}_{e\,y\,u\,w}\,\bar{\Omega}_x{}^e\,(d_2d_1) - \bar{R}_{p\,y\,u\,w}\,\bar{\Omega}_x{}^p\,(d_2d_1)$$

$$-\bar{R}_{x\,e\,y\,w}\,\bar{\Omega}_y{}^e(d_2d_1) - \bar{R}_{x\,p\,u\,w}\,\bar{\Omega}_y{}^p(d_2d_1) -$$

$$(4.6)\qquad -R_{x\,y\,e\,w}\,\bar{\Omega}_u{}^e(d_2d_1) - \bar{R}_{x\,y\,p\,w}\,\bar{\Omega}_u{}^p(d_2d_1)$$

$$-\bar{R}_{x\,y\,u\,e}\,\bar{\Omega}_w{}^e(d_2d_1) - \bar{R}_{x\,y\,u\,p}\,\bar{\Omega}_w{}^p(d_2d_1),$$

where each of the indices x,y,u,w belongs to one of the set {a,b,c,d,...} or {k,ℓ,m,n,...}; (4.4) takes the form

$$[D_1D_2]\ R_{\alpha\beta\gamma\delta} = [\bar{D}_1\bar{D}_2]\bar{R}_{abcd} B^{a\,b\,c\,d}_{\alpha\beta\gamma\delta} + [\bar{D}_1\bar{D}_2]\ R_{abcn} B^{a\,b\,c}_{\alpha\beta\gamma} N^n_\delta$$

$$[\bar{D}_1\bar{D}_2]\bar{R}_{abmd} B^{a\,b}_{\alpha\beta} N^m_\gamma B^d_\delta + [\bar{D}_1\bar{D}_2]\ \bar{R}_{a\ell cd} B^a_\alpha N^\ell_\beta B^{c\,d}_{\gamma\delta} +$$

$$[\bar{D}_1\bar{D}_2]\bar{R}_{kbcd} N^k_\alpha B^{b\,c\,d}_{\beta\gamma\delta} + [\bar{D}_1\bar{D}_2]\ \bar{R}_{abmn} B^{a\,b}_{\alpha\beta} N^{m\,n}_{\gamma\delta} +$$

$$[\bar{D}_1\bar{D}_2]\bar{R}_{a\ell cn} B^a_\alpha N^\ell_\beta B^c_\gamma N^n_\delta + [\bar{D}_1\bar{D}_2]\ \bar{R}_{kbcn} N^k_\alpha B^{b\,c}_{\beta\gamma} N^n_\delta +$$

$$[\bar{D}_1\bar{D}_2]\bar{R}_{a\ell md} B^a_\alpha N^{\ell\,m}_{\beta\gamma} B^d_\delta + [\bar{D}_1\bar{D}_2]\ \bar{R}_{kbnd} N^k_\alpha B^b_\beta N^n_\gamma B^d_\delta \tag{4.7}$$

$$[\bar{D}_1\bar{D}_2]\bar{R}_{k\ell cd} N^{k\,\ell}_{\alpha\beta} B^{c\,d}_{\gamma\delta} + [\bar{D}_1\bar{D}_2]\ \bar{R}_{a\ell mn} B^a_\alpha N^{\ell\ \ mn}_{\beta\gamma\delta} +$$

$$[\bar{D}_1\bar{D}_2]\bar{R}_{kbmn} N^k_\alpha B^b_\beta N^{m\,n}_{\gamma\delta} + [\bar{D}_1\bar{D}_2]\ \bar{R}_{k\ell cn} N^{k\,\ell}_{\alpha\beta} B^c_\gamma N^n_\delta +$$

$$[\bar{D}_1\bar{D}_2]\bar{R}_{k\ell md} N^{k\,\ell\,m}_{\alpha\beta\gamma} B^d_\delta + [\bar{D}_1\bar{D}_2]\bar{R}_{k\ell mn} N^{k\,\ell\,m\,n}_{\alpha\beta\gamma\delta}\ .$$

(4.7) is valid for the tensors P and S.

From (4.7), we obtain

$$[\bar{D}_1\bar{D}_2]\ \bar{R}_{abcd} = ([D_1D_2]\ R_{\alpha\beta\gamma\delta})\ B^{\alpha\,\beta\,\gamma\,\delta}_{abcd}$$

$$[\bar{D}_1\bar{D}_2]\ \bar{R}_{abcn} = ([D_1D_2]\ R_{\alpha\beta\gamma\delta})\ B^{\alpha\,\beta\,\gamma}_{abc} N^\delta_n \tag{4.8}$$

$$\vdots$$

$$[\bar{D}_1\bar{D}_2]\ \bar{R}_{k\ell mn} = ([D_1D_2]\ R_{\alpha\beta\gamma\delta})\ N^{\alpha\,\beta\,\gamma\,\delta}_{k\ell mn}$$

From (4.7) and (4.8), we obtain

THEOREM 4.1 The necessary and sufficient conditions that the Finsler space has a reccurent curvature tensor R of the secon(order i.e.

$$[D_1D_2]\ R_{\alpha\beta\gamma\delta} = K\ R_{\alpha\beta\gamma\delta} \tag{4.9}$$ are

$$[\bar{D}_1 \bar{D}_2]\, \bar{R}_{a\,b\,c\,d} = K\, \bar{R}_{a\,b\,c\,d}$$

(4.10) $$[\bar{D}_1 \bar{D}_2]\, \bar{R}_{a\,b\,c\,n} = K\, \bar{R}_{a\,b\,c\,n}$$

$$\vdots$$

$$[\bar{D}_1 \bar{D}_2]\, \bar{R}_{k\,\ell\,m\,n} = K\, \bar{R}_{k\,\ell\,m\,n}$$

The same theorem is true if everywhere in (4.9) and (4.10) instead of tensor R we put P or S with the same indices as those of R.

REFERENCES

1. H.Rund; Differential geometry of Finsler Spaces, Springer Verlag 1959.
2. H.Rund, The intrinsic and induced curvature theories of subspace of a Finsler space,Tensor v. 18, 1965.
3. Davies E.T., Subspaces of a Finsler space,Proc.London Math.Soc. (2) 49, 1945. 19-39.
4. O.Varga, Zur Differentialgeometrie der Hyperflachen in Finslerschen Räumen,Dtsch.Math 6. 1941 (192-212).
5. I.Čomić:Family of subspaces in a Finsler space,Tensor,Vol.37. 1982 (187-197).
6. I.Čomić: Family of subspaces in the Finsler space II (to appear in Matematički Vesnik, Beograd 1987).

REZIME

NEKE RELACIJE KOJE ZADOVOLJAVAJU TENZORI KRIVINA FINSLEROVOG PROSTORA

Ovde se posmatraju specijalni Finslerovi prostori u kojima postoje vektorska polja $B_a^\alpha(x)$ i $N_k(x)$ za koje važi $g_{\alpha\beta}(x\dot{x})\, B_a^\alpha(x)\, N_k^\alpha(x) = 0$. Dato je razlaganje alternisanog diferencijala nekog vektorskog polja kao i tenzora krivine u pravcu B_a^α i N_k^α.

Received by the editors September 2, 1986.