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SEQUENTIALLY CONDENSING MAPS

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ABSTRACT

A non-repulsive fixed point theorem is proved for upper semicontinuous, set-valued, acyclic and sequentially condensing transformations, T, of a convex, bounded, closed and infinite dimensional set, C, of a Banach space E into itself. It is also shown that in certain cases a sequentially condensing map can be made condensing with the choice of a suitable new measure of non-compactness.

1. INTRODUCTION

Let C be a convex, closed and bounded set of a Banach space E and let $T : C \to C$ be a single-valued or a set-valued map. It is known that if C is infinite dimensional and T is upper semicontinuous, condensing and such that T(x) is acyclic for every $x \in C$, then T has a non-repulsive fixed point (see [6], [3] and [1]).

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In the first part of this paper we show that the condition "T is condensing", i.e.

$$(1.1) \qquad \alpha(T(A)) < \alpha(A)$$

for every $A \subset C$, $\alpha(A) \neq 0$, can be relaxed by assuming that (1.1) holds only for countable sets $A \subset C$, $\alpha(A) \neq 0$.

In the second part we obtain a result which shows that at least in certain cases, the above generalization is only apparent, since we can introduce a new measure of non-compactness, β, such that

$$(1.2) \qquad \beta(T(A)) < \beta(A)$$

for every $A \subset C$, $\beta(A) \neq 0$.

Lemmas, Propositions and Theorems are stated and proved for set-valued maps, except in one case, (see Proposition 3.1), where the proof for single-valued maps is much simpler than the proof for set-valued maps (see Proposit. 3.2).

2. NOTATIONS AND DEFINITIONS

A set-valued map $T : C \to C$ is said to be *upper semicontinuous* if for every $x \in C$ the set $T(x)$ is compact; and for every neighborhood V of $T(x)$ there exists a neighborhood W of x such that

$$(2.1) \qquad T(y) \subset V$$

for every $y \in W$.

We say that T is *acyclic* if for every $x \in C$ the set $T(x)$ is acyclic in the Čech cohomology with rational coefficients.

The map T is said to be *condensing* if for every $A \subset C$ with $\alpha(A) \neq 0$ we have

(2.2) $$\alpha(T(A)) < \alpha(A)$$

where α is the Kuratowski measure of non-compactness [5], i.e.

(2.3) $$\alpha(A) = \mathit{inf}\{\varepsilon : \text{there is a finite covering of } A \text{ with sets whose diameter does not exceed } \varepsilon\}.$$

It is known that α has the following properties

(i) $\alpha(A) = 0$ iff $\bar{A}$ is compact;

(ii) $A \subset B$ $\alpha(A) \leq \alpha(B)$;

(iii) $\alpha(A \cup B) = max\{\alpha(A),\alpha(B)\}$;

(2.4) (iv) $\alpha(\overline{co}\ A) = \alpha(A)$ (see [2]), where $\overline{co}\ A$ is the closed convex hull of A;

(v) $\alpha(tA) = |t|\alpha(A)$;

(vi) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.

We shall say that $T : C \to C$ is *sequentially condensing* if (2.2) holds for countable sets $A \subset C$, $\alpha(A) \neq 0$. A condensing map T is sequentially condensing, but the converse may not be true. The advantage of replacing the previous condition "T is condensing" with the new one stems from the fact that the latter is more easily verifiable.

3. SEQUENTIALLY CONDENSING MAPS AND NON-REPULSIVE FIXED POINT THEOREMS

Lemma 3.1. *Let $f : C \to C$ be sequentially condensing and continuous. Then there exists a nonempty compact set K such that*

$$f(K) = K.$$

Proof. Let $x_0 \in C$. Consider the sequence

$$x_0, x_1 = f(x_0), \ldots, x_n = f(x_{n-1}), \ldots .$$

Define $A = \bigcup_{n=0}^{\infty} f^n(x_0)$, $f^0(x_0) = x_0$. Then A is compact (2.4.i) and $f(A) \subset A$. Hence (see [7]) there exists $K \subset \bar{A}$ such that $f(K) = K$.

Proposition 3.1. *Let* $f : C \to C$ *be sequentially condensing and continuous. Define* $C_\infty = \bigcap_{n=0}^{\infty} C_n$ *where* $C_0 = C$, $C_{n+1} = \overline{co}\, f(C_n)$.
Then C_∞ *is closed, convex and not empty.*

Proof. Note that $K \subset C_\infty$.

We now take up the multivalued case. We again want to show that C_∞ is not empty.

Proposition 3.2. *Let* $T : C \to C$ *be a sequentially condensing and upper semicontinuous map. There* C_∞ *is closed, convex and not empty.*

Proof. The only non-trivial property of C_∞ is that $C_\infty \neq \emptyset$. To prove it consider the set $A = \bigcup_{n=0}^{\infty} T^n(x_0)$ where $x_0 \in C$. Since the image of a compact set under an upper semicontinuous map is compact we see that A is separable. Let $\{y_n\} \subset \bar{A}$ be a dense sequence in $\bar{A}$ and let $\{x_n\} \subset A$ be such that $d(x_n, y_n) \to 0$ as $n \to +\infty$. Then for every $\varepsilon > 0$ there exists $n(\varepsilon)$ such that

$$\bigcup_n x_n \subset B(\bigcup_n y_n, \varepsilon) \cup \left(\bigcup_{i=1}^{n(\varepsilon)} x_i\right).$$

It follows that $\alpha(\{x_n\}) = \alpha(\{y_n\})$. Obviously $x_n \in T^p(x_0)$ for some p, i.e., there exists $w_n \in T^{p-1}(x_0)$ such that $x_n \in T(w_n)$. Assume $\alpha(A) > 0$. Then

$$\alpha(A) = \alpha(\overline{A}) = \alpha(\{y_n\}) = \alpha(\{x_n\}) \leq \alpha(\cup T(w_n)) <$$

$$< \alpha(\{w_n\}) \leq \alpha(A).$$

This contradiction shows that $\overline{A}$ is compact. This implies that if we select a sequence $z_0 = x_0$, $z_1 \in T(x_0)$, $z_2 \in T^2(x_0), \dots$; then there exists $n_p \to +\infty$ such that $z_{n_p} \to x$ for some $\overline{x} \in X$. From $z_{n_p} \in C_{n_p}$ and from $C_n \supset C_{n+1}$ we obtain $\overline{x} \in C_n$ for every n. Hence $C_\infty \neq \emptyset$.

Theorem 3.1. *Let* $T : C \to C$ *be sequentially condensing and upper semicontinuous. Then* C_∞ *is invariant and compact.*

Proof. Obviously $T(C_0) \subset C_0$. Moreover, from $T(C_n) \subset \subset C_n$ we can easily get $T(C_{n+1}) \subset C_{n+1}$. Hence C_∞ is invariant We also know that C_∞ is not empty. It remains to show that $\alpha(C_\infty) = 0$. We shall prove that for every sequence $\{x_n\} \subset C_\infty$ we have $\alpha(\{x_n\}) = 0$.

Let $Z = \{\{x_n\} : \exists\, n_p \to +\infty$ such that $x_n \in C_{n_p}\}$. Define

$$a = \sup\{\alpha(\{x_n\}) : \{x_n\} \in Z\} \tag{3.1}$$

and let $\{y_n\}_q \in Z$ be a sequence of elements in Z such that

$$\alpha(\{y_n\}_q) \to a \text{ as } q \to +\infty.$$

Let

$$M = \bigcup_q (\bigcup_n y_{nq}) \tag{3.2}$$

where y_{nq} is the n-th element of the q-th sequence. M is countable and obviously can be rearranged in a sequence $\{w_m\} \in Z$. Moreover, $\alpha(\{w_m\}) = a$.

Assume $a > 0$ and let $m_r \to +\infty$ be such that $w_m \in \overline{co}\, T(C_{m_r-1})$ for every m. Let $z_m \in co\, T(C_{m_2-1})$ be such that

$$\|w_m - z_m\| = \|y_m\| \leq \frac{1}{m} .$$

Then $\{y_n\}$ is convergent and $\alpha(\{z_m\}) = \alpha(\{w_m\})$. Since $z_m \in co\, T(C_{m_r-1})$ there exist finitely many points $x_{m_1},\ldots,x_{m_{j_m}}$ in C_{m_r-1} such that

$$z_m \in co(\bigcup_{i=0}^{j_m} T(x_{m_i})) .$$

Let

$$N = \bigcup(\bigcup_{i=0}^{j_m} x_{m_i}). \tag{3.3}$$

Then N is countable and it can be considered as an element of Z. Hence

$$a = \alpha(\{z_m\}) \leq \alpha(co(T(N))) = \alpha(T(N)) < \alpha(N) = a.$$

This contradiction shows that $a = 0$.

The basic ideas of the previous proof are derived from P. Massat [8].

We are now ready to prove the main result of this first part. Recall that a point $x_0 \in C$ such that $x_0 \in T(x_0)$ is said to be *repulsive* if there exists a neighborhood V of x_0 such that for every neighborhood W of x_0 there exists n such that

$$T^n(V \setminus W) \cap V = \emptyset. \tag{3.4}$$

This notion was first introduced by F. Browder [1].

Theorem 3.2. *Let* $T : C \to C$ *be an upper semicontinuous set-valued and sequentially condensing map. Assume that* T *is acyclic and dim* C = ∞. *Then* T *has a non-repulsive fixed point.*

Proof. Let $P \subset C$ be an infinite dimensional and compact set. Define

$$C_0 = C \cup P = C$$
$$C_n = \overline{co}(T(C_{n-1}) \cup P)$$
$$C_\infty = \bigcap_n C_n \quad .$$

Then C_∞ is non empty, closed, convex, infinite dimensional and invariant. Moreover, with a minor modification in the proof of Theorem 3.1 it can be shown that C_∞ is compact. Hence (see [3]) T has a non-repulsive fixed point in C_∞, which is obviously non-repulsive in C.

Remark 3.2. In the case of single valued, continuous maps Theorem 3.2 contains Sadovskii's [10] result as a particular case. In [10] the map f is assumed to be condensing and the principle of transfinite induction is used. Subsequently several authors (see [4], [7], [8], [9]) succeded in giving proofs of Sadovskii result without using the principle of transfinite induction. The first proofs of this type are due to M. Martelli [7] and M. Furi-A. Vignoli [4]. The following construction provides another proof which is noteworthy for its simplicity.

Let $x_0 \in C$. Define $A_0 = x_0$, $A_1 = \overline{co}(x_0 \cup f(x_0))$, $A_2 = \overline{co}(A_1 \cup f(A_1)), \ldots, A_n = \overline{co}(A_{n-1} \cup f(A_{n-1})), \ldots$. We see that

$$A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots$$

Therefore the set $A = \bigcup_n A_n$ is convex. Moreover $f(A) \subset A$, since $f(A_i) \subset A_{i+1}$. By the continuity of f we obtain $f(\overline{A}) \subset \overline{A}$. It remains to show that A is compact. This property of $\overline{A}$ is established by observing that

$$A \subset \overline{co}(x_o \cup f(A)).$$

Notice that the above construction cannot be extended to upper semicontinuous multivalued maps, T, since $T(A) \subset \subset A$ does not imply $T(\bar{A}) \subset \bar{A}$.

Remark 3.3. Properties (iii), (iv), and (vi) of the α-measure of noncompactness are never used in the Lemmas, Propositions and Theorems of this part.

4. β-MEASURE OF NON-COMPACTNESS

Let $T : C \to C$ be an upper semicontinuous set-valued map. Define the sequence of sets $\{C_n\}$ as in 3, and assume that $\alpha(C_n) \to 0$ as $n \to +\infty$. Under this assumption we want to show that we can define a new measure of non-compactness, β, such that $\beta(T(A)) < \beta(A)$ for every $A \subset C$, $\beta(A) \neq 0$.

We start by selecting a sequence $\{a_n\}$ of real numbers such that

$$a_n < a_{n+1}, \; a_o = 1, \; \lim_{n\to+\infty} a_n = 2.$$

Given $M \subset C$ we consider the sequence of sets

$$M_o = \overline{co}\, M, \; M_1 = \overline{co}\, T(M_o), \ldots$$

and we define

$$\beta(M) = max\{a_n \alpha(M_n) : n = 0,1,\ldots\}.$$

Let us now study the properties of β.

Proposition 4.1. *$\beta(M) = 0$ iff $\bar{M}$ is compact and* $\beta(co\, M) = \beta(M)$.

Proof. If $\beta(M) = 0$ then $\alpha(M) = 0$ and $\bar{M}$ is compact.

If $\overline{M}$ is compact then $T(M_0)$ is compact since T is upper semi-continuous. Therefore M_1 is compact, and, in general, M_n is compact for every n. Thus $\beta(M) = 0$.

The second property is evident.

It is also obvious that $M \subset N$ implies $\beta(M) < \beta(N)$ and if T is positively homogeneous then $\beta(tM)=t\beta(M)$ for $t>0$.

Theorem 4.1. *$T : C \to C$ is β-condensing, i.e. $\beta(T(M)) < \beta(M)$ for every non-compact set $\overline{M} \subset C$.*

Proof. Define $N = T(M)$. Then $N_0 = \overline{co}\, T(M) \subset \overline{co}\, T(M_0) = M_1$, and in general,

$$N_i \subset M_{i+1}$$

Since the sequence $\{a_n\}$ is increasing we have $\beta(M) = \max\{a_n\alpha(M_n): n = 0,1,\ldots\} > \max\{a_{n-1}\alpha(M_n) : n = 1,2,\ldots\} \geq \max\{a_n\alpha(N_n) : n = 0,1,\ldots\} = \beta(T(M))$.

Remark 4.1. We have presented our results in the context of Banach spaces, but it can be easily shown that they hold in complete, locally convex topological linear spaces, at least to the extent of proving that f has a fixed point. The definition of α needs to be suitably modified and the Tichonov fixed point theorem will be used once a compact, convex and invariant set C has been produced.

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REZIME

SEKVENCIJALNO KONDENZUJUĆA PRESLIKAVANJA

Nerepulzivna teorema o nepokretnoj tački je dokazana za od gore neprekidna, višeznačna, aciklična i sekvencijalno kondenzujuća preslikavanja T, konveksnog, ograničenog, zatvorenog i beskonačno dimenzionalnog skupa C u samog sebe, gde je $C \subset E$ a E je Banahov prostor. Pokazano je da u odredjenim slučajevima sekvencijalno kondenzujuća preslikavanja postaju kondenzujuća u odnosu na novu meru nekompaktnosti.

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