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COMMON FIXED POINTS OF TWO PAIRS OF WEAKLY COMMUTING MAPPINGS

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ABSTRACT

A common fixed point theorem is proved involving two pairs of weakly commuting mappings on a metric space (X,d) satisfying

$$d(Sx,Ty) \le g(d(Ix,Jy),d(Ix,Sx),d(Jy,Ty)), \text{ for all } x,y \text{ in } X$$

where $g : [0,\infty)^3 \to [0,\infty)$ satisfies (i) $g(1,1,1) = h < 1$ and (ii) if $u,v \ge 0$ and either $u \le g(u,v,v)$ or $u \le g(v,u,v)$ or $u \le g(v,v,u)$, then $u \le hv$.

1. THE FIXED POINT THEOREM

In the following, see [6], we defined two mappings S and I of a metric space (X,d) into itself to be weakly commuting if

$$d(SIx,ISx) \le d(Ix,Sx)$$

for all x in X. It is clear that two commuting mappings weakly commute but two weakly commuting mappings do not necessarily

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commute as is shown in Example 1 of [6].

The following theorem was proved in [3].

Theorem 1. *Let* S *and* I *be commuting mappings and let* T *and* J *be commuting mappings of a complete metric space* (X,d) *into itself satisfying the inequality*

(1) $$d(Sx,Ty) \le c \cdot \max\{d(Ix,Jy),d(Ix,Sx),d(Jy,Ty)\}$$

for all x, y *in* X, *where* $0 \le c < 1$. *If the range of* I *contains the range of* T *and the range of* J *contains the range of* S *and if one of* S, T, I *and* J *is continuous, then* S, T, I *and* J *have a unique common fixed point* z. *Further,* z *is the unique common fixed point of* S *and* I *and of* T *and* J.

Following Delbosco [1],we consider the set $\mathcal{S}$ of all real continuous functions $g : [0,\infty)^3 \to [0,\infty)$ satisfying the following properties:

(*i*) $g(1,1,1) = h < 1$,

(*ii*) let $u,v \ge 0$ be such that either $u \le g(u,v,v)$ or $u \le g(v,u,v)$ or $u \le g(v,v,u)$. Then $u \le hv$.

Delbosco [1] proved the following result.

Theorem 2. *Let* S *and* T *be two mappings of a complete metric space* (X,d) *into itself satisfying the inequality*

(2) $$d(Sx,Ty) \le g(d(x,y),d(x,Sx),d(y,Ty))$$

for all x, y *in* X, *where* g *is in* $\mathcal{S}$. *Then* S *and* T *have a unique common fixed point.*

We now unify and generalize Theorems 1 and 2 with the following result.

Theorem 3. *Let* S *and* I *be weakly commuting mappings and let* T *and* J *be weakly commuting mappings of a complete metric space* (X,d) *into itself satisfying the inequality*

$$(3) \qquad d(Sx,Ty) \le g(d(Ix,Jy),d(Ix,Sx),d(Jy,Ty))$$

for all x,y *in* X, *where* g *is in* S. *If the range of* I *contains the range of* T *and the range of* J *contains the range of* S *and if one of* S, T, I *and* J *is continuous, then* S, T, I *and* J *have a unique common fixed point* z. *Further,* z *is the unique common fixed point of* S *and* I *and of* T *and* J.

Proof. Let $x = x_0$ be an arbitrary point in X and let x_1 be a point such that $Sx_0 = Jx_1$. This can be done since the range of J contains the range of S. Let x_2 be a point such that $Tx_1 = Ix_2$. This can be done since the range of I contains the range of T. In general, we can choose x_{2n}, x_{2n+1} and x_{2n+2} such that $Sx_{2n} = Jx_{2n+1}$ and $Tx_{2n+1} = Ix_{2n+2}$ for $n = 0,1,2,\dots$.

Using inequality (3), we have

$$d(Sx_{2n},Tx_{2n+1}) \le g(d(Ix_{2n},Jx_{2n+1}),d(Ix_{2n},Sx_{2n}),d(Jx_{2n+1},Tx_{2n+1}))$$

$$\le g(d(Tx_{2n-1},Sx_{2n}),d(Tx_{2n-1},Sx_{2n}),d(Sx_{2n},Tx_{2n+1})),$$

which implies by property (ii)

$$d(Sx_{2n},Tx_{2n+1}) \le h\cdot d(Tx_{2n-1},Sx_{2n}).$$

Similarly,

$$d(Tx_{2n-1},Sx_{2n}) \le h\cdot d(Sx_{2n-2},Tx_{2n-1})$$

and so

$$d(Sx_{2n},Tx_{2n+1}) \le h\cdot d(Tx_{2n-1},Sx_{2n}) \le h^{2n}\cdot d(Sx_0,Tx_1)$$

for $n = 1,2,\ldots$. Since $h < 1$, we have that the sequence

$$(4) \qquad \{Sx_0, Tx_1, Sx_2, \ldots, Tx_{2n-1}, Sx_{2n}, Tx_{2n+1}, \ldots\}$$

is a Cauchy sequence in the complete metric space (X,d) and so has a limit z in X. Hence the sequences

$$\{Sx_{2n}\} = \{Jx_{2n+1}\} \text{ and } \{Tx_{2n-1}\} = \{Ix_{2n}\},$$

which are subsequences of (4), converge to the point z.

Let us now suppose that the mapping I is continuous, so that the sequences $\{I^2x_{2n}\}$ and $\{ISx_{2n}\}$ converge to the point Iz. Since S and I weakly commute, we have

$$d(SIx_{2n}, ISx_{2n}) \le d(Ix_{2n}, Sx_{2n})$$

and so the sequence $\{SIx_{2n}\}$ also converges to the point Iz. We now have

$$d(SIx_{2n}, Tx_{2n+1}) \le g(d(I^2x_{2n}, Jx_{2n+1}), d(I^2x_{2n}, SIx_{2n}), d(Jx_{2n+1}, Tx_{2n+1})).$$

Letting n tend to infinity and since g is continuous, we have

$$d(Iz,z) \le g(d(Iz,z),0,0) .$$

This implies by property (ii)

$$d(Iz,z) \le 0$$

and so

$$Iz = z.$$

Further

$$d(Sz,Tx_{2n+1}) \le g(d(Iz,Jx_{2n+1}),d(Iz,Sz),d(Jx_{2n+1},Tx_{2n+1}))$$

and letting n tend to infinity, we have

$$d(Sz,z) \le g(0,d(z,Sz),0)$$

which implies by property (ii)

$$Sz = z.$$

This means that z is in the range of S and since the range of J contains the range of S, there exists a point z' in X such that $Jz' = z$. Thus

$$d(z,Tz') = d(Sz,Tz') \le g(d(Iz,Jz'),d(Iz,Sz),d(Jz',Tz'))$$

$$= g(0,0,d(z,Tz')),$$

which implies by property (ii)

$$Tz' = z.$$

Since T and J weakly commute, we have

$$d(Tz,Jz) = d(TJz',JTz') \le d(Jz',Tz') = d(z,z) = 0.$$

Thus $Tz = Jz$ and so

$$d(z,Tz) = d(Sz,Tz) \le g(d(Iz,Jz),d(Iz,Sz),d(Jz,Tz))$$

$$= g(d(z,Tz),0,0),$$

which implies by property (ii)

$$z = Tz = Jz.$$

We have therefore proved that z is a common fixed point of S, T, I and J.

If the mapping J is continuous instead of I, then the proof that z is again a common fixed point of S, T, I and J is of course similar.

Now let us suppose that the mapping S is continuous, so that the sequence $\{S^2x_{2n}\}$ and $\{SIx_{2n}\}$ converge to the point Sz. Since S and I weakly commute, it follows as above that the sequence $\{ISx_{2n}\}$ also converges to the point Sz. Thus

$$d(S^2x_{2n},Tx_{2n+1}) \le g(d(ISx_{2n},Jx_{2n+1}),d(ISx_{2n},S^2x_{2n}), d(Jx_{2n+1},Tx_{2n+1})) .$$

Letting n tend to infinity, we have

$$d(Sz,z) \le g(d(Sz,z),0,0)$$

and so

$$Sz = z$$

by property (ii). Once again there exists a point z′ in X such that Jz′ = z. Thus

$$d(S^2x_{2n},Tz') \le g(d(ISx_{2n},Jz'),d(ISx_{2n},S^2x_{2n}),d(Jz',Tz')).$$

Letting n tend to infinity, it follows that

$$d(z,Tz') \le g(0,0,d(z,Tz'))$$

and so

$$Tz' = z$$

by property (ii). Since T and J weakly commute, it again follows as above that

$$Tz = Jz.$$

Further

$$d(Sx_{2n},Tz) \leq g(d(Ix_{2n},Jz),d(Ix_{2n},Sx_{2n}),d(Jz,Tz)).$$

Letting n tend to infinity, it follows that

$$d(z,Tz) \leq g(d(z,Tz),0,0)$$

and so

$$Tz = z = Jz$$

by property (ii). The point z is therefore in the range of T and since the range of I contains the range of T, there exists a point z'' in X such that $Iz'' = z$. Thus

$$d(Sz'',z) = d(Sz'',Tz) \leq g(d(Iz'',Jz),d(Iz'',Sz''),d(Jz,Tz))$$

$$= g(0,d(z,Sz''),0)$$

and so

$$Sz'' = z$$

by property (ii). Since S and I weakly commute, we have

$$d(Sz,Iz) = d(SIz'',ISz'') \leq d(Iz'',Sz'') = d(z,z) = 0.$$

Thus

$$Sz = Iz = z .$$

We have therefore proved once again that z is a common fixed point of S, T, I and J.

If the mapping T is continuous instead of S, then the proof that z is again a common fixed point of S, T, I and J is similar.

Now let w be a second common fixed point of S and I. Using inequality (3), we have

$$d(w,z) = d(Sw,Tz) \le g(d(Iw,Jz),d(Iw,Sw),d(Jz,Tz))$$

$$= g(d(w,z),0,0)$$

and it follows, from property (ii), that w = z. Then z is the unique common fixed point of S and I. Similarly, it is proved that z is the unique common fixed point of T and J. This completes the proof of the theorem.

Remark 1. Assuming $g(t_1,t_2,t_3) = c\cdot \max\{t_1,t_2,t_3\}$ for any $t_1,t_2,t_3 \ge 0$, it is easily seen that g is in $\mathcal{S}$. Then Theorem 1 follows from Theorem 3.

The following example shows that Theorem 3 is a stronger result than Theorem 1.

Example 1. Let X = {1,2,3,4} be a finite set with metric d defined by

$$d(1,2) = 3, \quad d(1,3) = 7.9, \quad d(1,4) = 7.95,$$
$$d(2,3) = 6, \quad d(2,4) = 8, \quad d(3,4) = 12.$$

Let I = J be the identity on X and define S and T by

$$S1 = S3 = S4 = 3, \quad S2 = 4, \quad T1 = T3 = T2 = 3, \quad T4 = 1.$$

A routine calculation shows that Theorem 3 is satisfied if one assumes that

$$g(t_1,t_2,t_3) = hk^{(t_1-t_2)^2(t_2-t_3)^2(t_3-t_1)^2}\cdot \max\{t_1,t_2,t_3\},$$

where h and k are real numbers such that

$$\frac{7.95}{8} \le h < 1, \quad k > 1.$$

However, inequality (1) does not hold since we have for x = 2 and y = 3

$$d(S2,T3) = d(4,3) = 12 > 8 = max\{6,8,0\}$$

$$= max\{ d(2,3),d(2,S2),d(3,T3)\}.$$

Remark 2. If I = J is the identity on X, Theorem 3 becomes Theorem 2.

Remark 3. We refer to the examples of [3], where it is shown that the weak commutativity (see [7]) of T and J, the range of I contains the range of T and the continuity of one of the mappings S, T, I and J are necessary conditions in Theorem 1 and therefore also in Theorem 3.

2. A COMPARISON

The authors of [4], generalizing Theorem 1, considered the family $\mathcal{F}$ of all real functions $f : [0,\infty) \to [0,\infty)$ such that f is increasing, continuous from the right and $f(t) < t$ for any $t > 0$. In particular, they established, under suitable assumptions for the mappings S, T, I and J, a similar result to Theorem 3 using the following inequality:

$$(5) \quad d(Sx,Ty) \le f(max\{d(Ix,Jy),d(Ix,Sx),d(Jy,Ty),d(Ix,Ty),\\ d(Jy,Sx)\})$$

for all x,y in X, where f is in $\mathcal{F}$. Of course, this result extends the results of [2] and [7], where the authors considered analogous contractive conditions by using a function f in $\mathcal{F}$. As pointed out by Delbosco [1], a function g in *S* generally does

not belong to $\mathcal{F}$ and vice versa, i.e. inequality (3) and inequality (5) are two different generalizations of inequality (1).
We illustrate this with suitable examples.

Example 2. Let X, S, T, I, J and d be as in Example 1. We already know that inequality (3) holds. We now show that inequality (5) is not satisfied. Indeed, we have for $x = 2$, $y = 3$ and for any f in $\mathcal{F}$

$$d(S2,T3) = d(4,3) = 12 > f(12) = f(\max\{6,8,0,6,12\})$$
$$= f(\max\{d(I2,J3),d(S2,J3),d(I2,S2),d(J3,S3),d(I2,T3)\}) .$$

Adopting the same technical proof of Theorem 3, it is not difficult to prove the following result.

Theorem 4. *Let* S *and* I *be weakly commuting mappings and let* T *and* J *be weakly commuting mappings of a complete metric space* (X,d) *into itself satisfying the inequality*

$$(6)\quad d(Sx,Ty) \le \max\{cd(Ix,Jy),cd(Ix,Sx),cd(Jy,Ty),ad(Ix,Ty) + bd(Jy,Sx)\}$$

for all x,y *in* X, *where* a, b, c *are real numbers such that* $0 \le c < 1$, $0 \le a + b < 1$ *and*

$$(7)\qquad c\cdot\max\{a/(1-a),b/(1-b)\} < 1.$$

If the range of I *contains the range of* T *and the range of* J *contains the range of* S *and if one of* S, T, I *and* J *is continuous, then* S, T, I *and* J *have a unique common fixed point* z. *Further,* z *is the unique common fixed point of* S *and* I *and of* T *and* J.

Proof. Let $x = x_0$ be an arbitrary point in X and, as in the proof of Theorem 3, we consider the sequence (4) such that $Sx_{2n} = Jx_{2n+1}$ and $Tx_{2n+1} = Ix_{2n+2}$ for $n = 0,1,2,\ldots$.

Using inequality (6), we have

$$d(Sx_{2n},Tx_{2n+1}) \le \max\{cd(Ix_{2n},Jx_{2n+1}),cd(Ix_{2n},Sx_{2n}),$$
$$cd(Jx_{2n+1},Tx_{2n+1}),ad(Ix_{2n},Tx_{2n+1}) + bd(Jx_{2n+1},Sx_{2n})\}$$
$$\le \max\{cd(Tx_{2n-1},Sx_{2n}),cd(Sx_{2n},Tx_{2n+1}),$$
$$a[d(Tx_{2n-1},Sx_{2n}) + d(Sx_{2n},Tx_{2n+1})]\}$$

which implies that

$$d(Sx_{2n},Tx_{2n+1}) \le \max\{c,a/(1-a)\}\cdot d(Tx_{2n-1},Sx_{2n}).$$

Similarly

$$d(Tx_{2n-1},Sx_{2n}) \le \max\{c,b/(1-b)\}\cdot d(Sx_{2n-2},Tx_{2n-1})$$

and so

(8) $$d(Sx_{2n},Tx_{2n+1}) \le \alpha^n\cdot d(Sx_0,Tx_1)$$

and

(9) $$d(Tx_{2n-1},Sx_{2n}) \le \alpha^{n-1}\, d(Tx_1,Sx_2),$$

where

$$\alpha = \max\{c,a/(1-a)\}\cdot\max\{c,b/(1-b)\}.$$

It is easily seen that $0 \le \alpha < 1$. This implies, by (8) and (9), that the sequence (4) is a Cauchy sequence in the complete metric space X and so has a limit z. From now on, the proof is similar to the proof of Theorem 3 and we therefore omit it.

Remark 4. If I = J is the identity on X, Theorem 4 becomes Theorem 6 of [5].

Remark 5. Assuming $f(t) = \max\{ct, at + bt\}$ for any $t > 0$ with $0 \le c < 1$ and $0 \le a + b < 1$, it is immediately seen that f is in $\mathcal{F}$. Theorem 4 can therefore also be obtained as a consequence of Theorem 1 of [4].

Since inequality (6) is a particular case of inequality (5), we now give example where inequality (6) is satisfied but inequality (3) does not hold.

Example 3. Let $X = \{1,2,3,4\}$ be a finite set with metric d defined by

$$d(1,3) = d(1,4) = d(2,3) = d(2,4) = 1,$$

$$d(1,2) = d(3,4) = 2.$$

Let I be the identity on X and define S, T, J on X by

$$S1 = 2, \quad S2 = S3 = 1, \quad S4 = 3,$$

$$T1 = T2 = T3 = T4 = 4,$$

$$J1 = 2, \; J2 = 1, \quad J3 = 3, \quad J4 = 4.$$

All the conditions of Theorem 4 are satisfied since

$$S(X) = \{1,2,4\} \subset X = J(X),$$

$$T(X) = \{4\} \subset X = I(X)$$

and of course S, T, I and J are continuous. Further, it is easily seen that S and T commute with I and J respectively and inequalities (6) and (7) hold with $a = 1/2$, $b = 0$ and $c = 1/2$. Inequality (3) is not satisfied. Indeed, we have for $x = 3$, $y = 1$ and any g in $\mathcal{G}$:

$$d(S3,T1) = d(1,4) = 1 > g(1,1,1)$$

$$= g(d(I3,J1),d(I3,S3),d(J1,T1)).$$

We now observe that inequality (6) implies

$$d(Sx,Ty) \le \max\{cd(Ix,Jy),cd(Ix,Sx),cd(Jy,Ty),$$
$$(a+b)\cdot\max\{d(Ix,Ty),d(Jy,Sx)\}\}$$

for all x,y in X and so

$$(10) \qquad d(Sx,Ty) \le \beta\cdot\max\{d(Ix,Jy),d(Ix,Sx),d(Jy,Ty),$$
$$d(Ix,Ty),d(Jy,Sx)\}$$

for all x,y in X, where $0 \le \beta = \max\{c,a+b\} < 1$.

We now show that Theorem 4 fails if we assume the more general inequality (10) instead of inequality (6) when $1/2 \le \beta < 1$. To see this, we give the following example mainly inspired by Example 6 of [5].

Example 4. $X = \{1,2,3,4\}$ be a finite set with metric d defined by

$$d(1,2) = d(3,4) = 2, \quad d(1,3) = d(2,4) = 1,$$
$$d(1,4) = d(2,3) = 3/2.$$

Let I be the identity on X and define S, T, J on X by

$$S1 = S4 = 2, \quad S2 = S3 = 1,$$
$$T1 = T3 = 4, \quad T2 = T4 = 3,$$
$$J1 = 3, \quad J2 = 4, \quad J3 = 1, \quad J4 = 2.$$

We have

$$S(X) = \{1,2\} \subset X = J(X)$$
$$T(X) = \{3,4\} \subset X = I(X)$$

and of course S, T, I and J are continuous. Further,

$$d(TJ1,JT1) = d(T3,J4) = d(4,2) = 1 < 2 = d(3,4) = d(J1,T1),$$

$$d(TJ2,JT2) = d(T4,J3) = d(3,1) = 1 < 2 = d(4,3) = d(J2,T2),$$

$$d(TJ3,JT3) = d(T1,J4) = d(4,2) = 1 < 3/2 = d(1,4) = d(J3,T3),$$

$$d(TJ4,JT4) = d(T2,J3) = d(3,1) = 1 < 3/2 = d(2,3) = d(J4,T4).$$

Thus T weakly commutes with J and S commutes with I. It is easily seen that inequality (10) is satisfied with $1/2 \leq \beta < 1$, but S, T, I and J do not have common fixed points.

We conclude observing that if $0 \leq \beta < 1/2$, then inequality (10) implies

$$d(Sx,Ty) \leq \max\{\beta d(Ix,Jy), \beta d(Ix,Sx), \beta d(Jy,Ty), \beta d(Ix,Ty) + \beta d(Jy,Sx)\}$$

for all x,y in X. This inequality is formally analogous to (6) by putting $\beta = a = b = c$. Therefore Theorem 4 remains valid if inequality (6) is replaced by the equivalent inequality (10), provided that $0 \leq \beta < 1/2$. In this case we note that inequality (7) is also satisfied, since $\beta < 1/2$ implies that

$$\beta \frac{\beta}{1-\beta} < \frac{1}{2} < 1.$$

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REZIME

ZAJEDNIČKE NEPOKRETNE TAČKE DVA PARA SLABO KOMUTATIVNIH PRESLIKAVANJA

Dokazana je teorema o zajedničkoj nepokretnoj tački za dva para slabo komutativnih preslikavanja definisanih nad metričkim prostorom (X,d), koji zadovoljavaju:

$$d(Sx,Ty) \le g(d(Ix,Jy),d(Ix,Sx),d(Jy,Ty)), \text{ za sve } x,y \text{ u } X$$

gde $g: [0,\infty)^3 \to [0,\infty)$ zadovoljava:

(i) $g(1,1,1) = h < 1$

(ii) ako je $u,v \ge 0$ i ili $u \le g(u,v,v)$ ili $u \le g(v,u,v)$ ili $u \le g(v,v,u)$ tada je $u \le hv$.

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