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SOME FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN BANACH SPACES AND THEIR APPLICATIONS

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ABSTRACT

In this paper a multivalued generalization of the well known Melvin fixed point theorem [16] is obtained. Using this generalization an existence result for a class of functional equations is obtained.

INTRODUCTION

For functional-differential equations a tool that has been widely used to prove existence of solutions has been the fixed point method. The aim of this paper is to obtain multivalued generalizations of the well known Melvin fixed point theorem [16] and to apply these results on the existence problem for functional equations of the form:

(1) $$g(t)\in f(t,(Tg)(t),\int_0^t g(u)du) \qquad t\in I=[0,a].$$

An existence result for the equation $x(t)\in F(t,x)$ is also obtained and may be used in the proof of the existence of a solution of functional equations of the form:

$$x(t)\in f(t,\int_0^{s(t)} g(t,u,x(u))du, x(r(t))), t\in I$$

where $r:I\to I$ and $s:I\to I$ are continuous mappings.

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Some multivalued generalizations of Melvin's fixed point theorem are proved in [10], [11], [12], [20] and [21].

In [16] Melvin proved the following fixed point theorem:

Let (X,d) be a metric space, A a nonempty, closed and convex subset of a Banach space $(Y,\|\ \|)$, $T:A\to X$ a continuous mapping such that $\overline{T(A)}$ is compact in X and $G:\overline{T(A)}\times A\to A$ so that the following conditions are satisfied:

(i) $G(\cdot,u):\overline{T(A)}\to A$ is continuous, for each fixed $u\in A$.

(ii) $\|G(x,u)-G(x,v)\| \leq q\|u-v\|$, for each $x\in\overline{T(A)}$

and each $u,v\in A$, where $q\in[0,1)$.

Then there exists $x\in A$ such that $x = G(Tx,x)$.

The idea of the proof of the above fixed point theorem is the following. First, the author proved the existence of the continuous mapping $f:\overline{T(A)}\to A$ such that for every $x\in\overline{T(A)}$ $f(x) = G(x,f(x))$ and then, applying the Schauder fixed point theorem to the mapping Q defined by $Qx = f(Tx)$ $(x\in A)$, the assertion in the above theorem follows.

Hence, the main problem in the multivalued case of the mapping G is to prove the existence of a continuous mapping $f:\overline{T(A)}\to A$ such that for every $x\in\overline{T(A)}$: $f(x)\in G(x,f(x))$.
If we denote by F the multivalued mapping defined by:

$$F(u) = \{x \mid x\in A, x\in G(u,x)\}\quad (u\in\overline{T(A)})$$

it is obvious that the mapping f (if it exists) is, in fact, a continuous selector of the mapping F. Let us remark that the upper semicontinuity of the mapping F is investigated in Cheng's paper [3].

Using a result of Banks and Jacobs, which holds only for uniformly convex Banach spaces, in [12] the following theorem is obtained:

THEOREM A Let X be a metric space, A a nonempty, closed and convex subset of a uniformly convex Banach space $(Y,\|\ \|)$, $T:A\to X$ a continuous mapping such that $T(A)$ is compact in X and $G:T(A)\times A\to CCB(A)$ (the family of all nonempty, closed convex and bounded subsets of A) such that the following conditions are satisfied:

1. $G(\cdot,u):\overline{T(A)}\to CCB(A)$ <u>is continuous for each fixed</u> $u\in A$.
2. <u>If</u> D <u>is the Hausdorff metric in</u> Y <u>then</u>:

$$D(G(x,u),G(x,v))\le h(\|u-v\|)$$

<u>for each</u> $x\in T(A)$ <u>and</u> $u,v\in A$ <u>where</u> $h:[0,\infty)\to[0,\infty)$ <u>is a nondecreasing mapping such that</u> $\sum_{n\in N} h^n(t)<\infty\ (t>0)$.

<u>Then there exists</u> $x\in A$ <u>such that</u> $x\in G(Tx,x)$.

If $h(t)=kt\ (t\ge 0)$, where $k\in(0,1)$ and Y is a Hilbert space Theorem A is proved by Kisielewicz in [11] and if Y is a Banach space by Rzepecki in [21]. A generalization of Theorem A is given in [20].

Using an another approach (Michael's selection theorem) we prove a multivalued generalization of Melvin's fixed point theorem, where Y is an arbitrary Banach space and the mapping h satisfies some different conditions than in Theorem A.

Theorem 1 of this paper can be generalized to the case of a complete paranormed space Y (such a space is $S(I,R^n)$ - the space of all classes of Lebesgue measurable functions from $I=[0,a]$ into R^n) by using Zima's fixed point theorem and a generalization of Michael's selection theorem recently obtained by the authors and Lj.Gajić in [10].

1. PRELIMINARIES

Let $h:[0,\infty)\to[0,\infty)$ and introduce the following conditions as in [28]:

(A_1) h is right continuous.

(A_2) For each $q>0$:

$$\sup\{u \mid u\le h(u)+q\}=m(q)<\infty.$$

(A_3) For each $q>0$: $\lim_{n\to\infty} h^n(m(q))=0$.

(A_4) For any $u\ge 0$ and $v\ge 0$: $h(u)+h(v)\le h(u+v)$.

From [28] we have the following fixed point theorem.

THEOREM B <u>Suppose that for every</u> $x,y\in X$, <u>where</u> (X,d) <u>is a complete metric space, the following condition holds:</u>

$$D(Tx,Ty)\le h(d(x,y))$$

where $T:X\to CB(X)$ (the family of all nonempty, closed and bounded subsets of X) is continuous and h satisfies (A_1)-(A_4). Then there exists $x\in X$ such that $x\in Tx$.

Remark: It is obvious that (A_4) implies that h is a nondecreasing mapping . Further, if h is a nondecreasing mapping such that $\lim_{n\to\infty} h^n(q)=0$, for each $q>0$ and $\lim_{q\to\infty}(q-h(q))=\infty$ the condition (A_2) holds [18] . Condition (A_4) is satisfied, for example , for $h(t)=q(t)t$, where q is a nondecreasing mapping .

We shall also use in the next text the following fixed point theorem [19] , [24] .

THEOREM C Let (X,d) be a complete metric space and $T:X\to CB(X)$ so that for every $x,y\in X$:

$$D(Tx,Ty)\leq q(d(x,y))d(x,y)$$

where $q:[0,\infty)\to[0,1)$ is upper semicontinuous from the right . Then there exists $x\in X$ such that $x\in Tx$.

Let E be a linear space over the real or complex number field . The function $q:E\to[0,\infty)$ is said to be a paranorm if and only if :

1. $q(x)=0 \Longleftrightarrow x=0$.
2. $q(-x)=q(x)$, for every $x\in E$.
3. $q(x+y)\leq q(x)+q(y)$, for every $x,y\in E$.
4. If $q(x_n-x_o)\to 0$ and $r_n\to r_o$ (r_n,r_o from the scalar field) then $q(r_nx_n-r_ox_o)\to 0$.

Then (E,q) is a paranormed space, which is also a topological vector space in which the family of neighbourhoods of zero is given by $V=\{V_r\}_{r>0}$, where $V_r=\{x\mid x\in E, q(x)<r\}$. In [27] K.Zima proved a generalization of Schauder's fixed point theorem for mapping $T:K\to K(K\subset E)$, where K satisfies the following condition (in the next text the Zima condition) :

$$q(rx)\leq C(K)\,r\,q(x) \quad \text{, for every } r\in[0,1]$$

and every $x\in K-K$. Here, $C(K)$ is a constant which depends on K .

Let us give an example of a subset which satisfies the Zima condition .

Let $a>0$, $I=[0,a]$ and for $x\in S(I,R^n)$ (the space of all equivalence classes of Lebesgue measurable functions, defined on I with values in R^n) , $q(x)$ is defined by :

$$q(x) = \int_I \frac{|x(t)|_{R^n}}{1+|x(t)|_{R^n}}dt \quad , \{x(t)\}\in x .$$

It is easy to see that q is a paranorm . Let $s>0$ and $y\in S(I,R^n)$. Further ,let $K = y + K_s$,where K_s is defined by :

$$K_s = \{x \mid x\in S(I,R^n) , |x(t)|_{R^n} \leq s ,t\in I\} .$$

Then [8] we have that :

$$q(r(x_1-x_2)) \leq (1+2s)r\, q(x_1-x_2)$$

for every $x_1,x_2\in K_s$ and every $r\geq 0$. This means that $C(K_s) = 1 + 2s$ and so $C(K) = 1 + 2s$. Hence $C(K)=C(K_s) = 1+2s$.

REMARK: Let E be a topological vector space, $\mathcal{U}$ the family of all neighbourhoods of zero in E and K a nonempty subset of E . Motivated by Zima's condition ,in [8] we introduced the following definition: A subset K of E is of <u>Zima's type</u> if and only of for every $V\in\mathcal{U}$ there exists $U\in\mathcal{U}$ so that $co(U\cap(K-K))\subset V$ (co is the operator of the convex hull) .

It is easy to see that if K satisfies the Zima condition it is of Zima's type . It is proved in [8] that every subset of Zima's belongs to the class of admissible subsets of a topological vector space [9] . The class of subsets of Zima's type has many important properties which are useful in the fixed point theory [9] .

THEOREM D <u>Let X be a paracompact topological space, (Y,q) a complete paranormed space, $F:X\to CC(Y)$ (the family of all convex closed nonempty subsets of Y) a lower semicontinuous mapping such that $K = co\,F(X)$ (the convex hull of F(X)) satisfies the Zima condition . Then there exists a continuous selection of the mapping F.</u>

2. EXISTENCE RESULTS FOR THE EQUATION $x \in G(Tx,x)$

First, we shall prove the following lemma.

LEMMA 1 Let (X,d) be a metric space, A a nonempty closed and convex subset of a Banach space $(Y, \| \ \|)$, $T: A \to X$ a continuous mapping such that $\overline{T(A)}$ is compact in X and $G: \overline{T(A)} \times A \to CCB(A)$ is such that the following conditions are satisfied:

(i) $G(\cdot,u): \overline{T(A)} \to CCB(A)$ is continuous, for each fixed $u \in A$.

(ii) $D(G(x,u),G(x,v)) \leq h(\|u-v\|)$, for each $x \in \overline{T(A)}$ and $u \in A, v \in A$, where $h: [0,\infty) \to [0,\infty)$ satisfies $(A_1),(A_2),(A_4)$ and $\lim_{n\to\infty} h^n(q) = 0$, for every $q > 0$.

Then there exists a continuous mapping $f: \overline{T(A)} \to A$ so that

$$f(x) \in G(x,f(x)), \text{ for every } x \in \overline{T(A)}.$$

Proof: Let $U = \{g \mid g: \overline{T(A)} \to Y, g \text{ is continuous}\}$ and for every $g_1, g_2 \in U$ let:

$$r(g_1,g_2) = \sup_{x \in \overline{T(A)}} \|g_1(x) - g_2(x)\|.$$

Then (U,r) is a complete metric space and let the mapping F be defined in the following way:

$$F(g) = \{u \mid u \in U, u(x) \in G(x,g(x)), x \in \overline{T(A)}\}, \text{ for } g \in U.$$

As in [12] (Lemma 2, which holds for any Banach space Y) it follows that the mapping $x \mapsto G(x,g(x))$ $(x \in \overline{T(A)})$ is continuous for every $g \in U$ and so from Michael's selection theorem [17] it follows that $F(g) \neq \emptyset$, for every $g \in U$. Let us prove that for every $g_1, g_2 \in U$ we have:

(2) $$D_r(F(g_1),F(g_2)) \leq h(r(g_1,g_2))$$

where D_r is the Hausdorff metric induced by r.

In order to prove (2) we shall prove that for every $v \in F(g_1)$:

(3) $$d_r(v,F(g_2)) \leq h(r(g_1,g_2))$$

and for every $s \in F(g_2)$:

(4) $$d_r(F(g_1),s) \leq h(r(g_1,g_2)).$$

Here $d_r(v,F(g_2)) = \inf_{z\in F(g_2)} r(v,z)$. Let $t>0$. We shall prove the existence of $w\in F(g_2)$ so that $r(v,w)< h(r(g_1,g_2))+t$ which implies (3).

From the definition of the Hausdorff metric D it follows that for every $x\in\overline{T(A)}$ there exists $y_x\in A$ such that $y_x\in G(x,g_2(x))$ and:

(5) $$\|v(x)-y_x\| < D(G(x,g_1(x)),G(x,g_2(x)))+ t/2 .$$

From (ii) and (5) it follows that:

$$\|v(x)-y_x\| < h(\|g_1(x)-g_2(x)\|)+t/2$$

and since h is nondecreasing and $\|g_1(x)-g_2(x)\| \leq r(g_1,g_2)$ we obtain that :

$$\|v(x)-y_x\|< h(r(g_1,g_2)) + t/2$$

From Michael selection theorem and Example 1.3 [17] it follows that for every $x\in\overline{T(A)}$ there exists a selector of the mapping $z\to G(z,g_2(z))(z\in\overline{T(A)})$, denoted by $w_x:\overline{T(A)}\to A$ such that $w_x(x)=y_x$. Since w_x is a selector of the mapping $z\to G(z,g_2(z))$ we have that $w_x(z)\in G(z,g_2(z))$,for every $z\in\overline{T(A)}$. Since v and w are continuous ,there exists an open neighbourhood U_x of x so that

(6) $$\|v(z)-w_x(z)\|< h(r(g_1,g_2)) + t/2$$

for every $z\in U_x$.The set $\overline{T(A)}$ is paracompact(as a metric space) and s there exists a locally finite partition of unity P subordinated to covering $\{U_x\}_{x\in\overline{T(A)}}$. Let $P=\{p_s\}_{s\in A}$ and suppose that $\mathrm{supp}\, p_s\subset U_{x_s}$. This means that $p_s(x)\neq 0$ implies that $x\in U_{x_s}$.

Let $w:\overline{T(A)}\to A$ be defined in the following way:

$$w(z) = \sum_{s\in A} p_s(z)w_{x_s}(z) \quad ,z\in\overline{T(A)} .$$

Since p_s and w_{x_s} are continuous and P is a locally finite partition of unity it follows that w is continuous and from $w_{x_s}(z)\in G(z,g_2(z)),z\in\overline{T(A)}$ and the convexity of $G(z,g_2(z))$ it follows that $w(z)\in G(z,g_2(z)),z\in\overline{T(A)}$.

Let us prove that for every $z\in\overline{T(A)}$:

$$\|v(z)-w(z)\|< h(r(g_1,g_2)) + t/2 .$$

We have that :

$$\|v(z)-w(z)\| = \Big\|\sum_{s\in A} p_s(z)v(z)-\sum_{s\in A} p_s(z)w_{x_s}(z)\Big\| =$$

$$= \Big\|\sum_{s\in A} p_s(z)\,[v(z)-w_{x_s}(z)]\Big\| =$$

$$=\| \sum_{s:p_s(z)\neq 0} p_s(z)\,[v(z)-w_{x_s}(z)]\,\|$$

$$< \sum_{s:p_s(z)\neq 0} p_s(z)\,\|v(z)-w_{x_s}(z)\| \quad .$$

Since from $p_s(z)\neq 0$ we obtain that $z\in U_{x_s}$ it follows that $p_s(z)\neq 0$ implies that $\|v(z)-w_{x_s}(z)\|< h(r(g_1,g_2))+t/2$ and so (6) implies that:

$$\|v(z)-w(z)\|< \sum_{s:p_s(z)\neq 0} p_s(z)\,[h(r(g_1,g_2)) + t/2\,]$$

$$=h(r(g_1,g_2))+ t/2 \ .$$

Hence we have that :

$$r(v,w)\leq h(r(g_1,g_2)) + t/2< h(r(g_1,g_2)) + t$$

and so (3) is proved . Similarly we can prove (4) .

From Theorem B it follows that there exists $f\in U$ so that $F(f)$ contains f which means that $f(x)\in G(x,f(x))$,for every $x\in\overline{T(A)}$.

Remark: A similar method as in the proof of Lemma 1 is used in the proof of Theorem 25.2 in [4] . If $h(t) = q(t)t$, where q is a nondecreasing and upper semicontinuous from the right function from $[0,\infty)$ into $[0,1)$ we can apply Theorem C in order to prove the existence of $f\in U$ such that $f\in F(f)$. If $h:[0,\infty)\to[0,\infty)$ is upper semicontinuous and strictly increasing function such that $\sum_{n\in N} h^n(t)<\infty$ $(t> 0)$ from Corollary 2 [25] it follows the existence of $f\in U$ such that $f\in F(f)$. Further,if Y is a uniformly convex Banach space and $h(t)=q(t)t$ $(t >0)$,where $q:[0,\infty)\to[0,1)$ it is enough to suppose that q is only a nondecreasing function . Namely , in this case $h^n(t)\leq (q(t))^n t$ and so $\sum_{n\in N} h^n(t)<\infty$ and we can apply the Lemma from [12] .

THEOREM 1 Let all the conditions of Lemma 1 be satisfied. Then there exists $x\in A$ so that $x\in G(T(x),x)$.

Proof: Let $Qx = f(Tx)$,$x\in A$,where $f\in U$ is from Lemma 1 . From the continuity of f and the compactness of $\overline{T(A)}$ it follows that the mapping $Q:A\to A$ satisfies all the conditions of the

Schauder fixed point theorem and so there exists $x_o \in A$ such that $x_o \in Qx_o$. Hence, $x_o \in G(T(x_o),x_o)$.

LEMMA 2. Let $(Y,\|\ \|)$ be a Banach space, A a nonempty, closed and convex subset of Y, M a metric space, $G: M\times A \to CCB(A)$, $h: [0,\infty)\to[0,\infty)$ a nondecreasing mapping such that $h(t)< t$ $(t>0)$ and so that the following conditions are satisfied:

1. For every $y\in A$, $x\mapsto G(x,y)\ (x\in M)$ is a continuous mapping.
2. There exist $y_o \in A$, $r>1$ and $t > \sup\{ d(y_o,G(x,y_o)) | x\in M \}$ so that the sequence $\{t_n\}_{n\in N\cup\{0\}}$ defined by:

$$t_{n+1}=t_n+h(r(t_n-t_{n-1})) \ ,n\in N \quad ,t_o=0,t_1=t$$

is bounded.

3. For every $x\in M$ and $y_i \in A\ (i\in\{1,2\})$:

$$D(G(x,y_1),G(x,y_2)) \leq h(\| y_1-y_2\|) \quad .$$

If M is compact or A is bounded then there exists a continuous mapping $f:M\to A$ such that $f(x)\in G(x,f(x))$, $x\in M$.

Proof: Since $r>1$ it follows from Lemma 7.1 [17] that there exists $f_1:M\to A$ which is continuous and such that :

$$\| f_1(x)-y_o\| = r\, d(y_o,G(x,y_o)) \text{ and } f_1(x)\in G(x,y_o)$$

for every $x\in M$. Let $y_o = f_o(x)$, for every $x\in M$.

It is easy to see that for every $n\in N$ there exists a continuous mapping $f_n:M\to A$, such that $f_n(x)\in G(x,f_{n-1}(x))$, for every $x\in M$ and :

$$\| f_n(x)-f_{n-1}(x)\| = r\, d(f_{n-1}(x),G(x,f_{n-1}(x))) \quad .$$

Let us prove that for every $n\in N$:

$$\| f_n(x)-f_{n-1}(x)\| \leq r(t_n-t_{n-1}) \text{ for every } x\in M \, .$$

For $n = 1$ we have :

$$\| f_1(x)-f_o(x)\| = \| f_1(x)-y_o\| = r\, d(y_o,G(x,y_o))$$

$$\leq r \sup_{x\in M} d(y_o,G(x,y_o)) \leq r(t_1-t_o) \quad .$$

Suppose that $\| f_n(x)-f_{n-1}(x)\| \leq r(t_n-t_{n-1})$. Then

$$\| f_{n+1}(x)-f_n(x)\| \leq rD(G(x,f_n(x)),G(x,f_{n-1}(x))) \leq$$

$$\leq r\, h(\| f_n(x)-f_{n-1}(x)\|) \leq r\, h(r(t_n-t_{n-1})) = r(t_{n+1}-t_n) \quad .$$

From this we obtain that:

$$\| f_{n+m}(x)-f_n(x)\| \leq \sum_{k=n+1}^{n+m} \| f_k(x)-f_{k-1}(x)\| \leq r \sum_{k=n+1}^{n+m} (t_k-t_{k-1})$$

$$= r(t_{n+m}-t_n) \quad .$$

Since $\{t_n\}_{n\in N\cup\{0\}}$ is a monotone nondecreasing sequence such that $t_o=0<t_1<t_2<\ldots<a$, it follows that there exists t^* such that $\lim_{n\to\infty} t_n = t^*$. Hence f_n is a uniformly convergent sequence of functions and let $f(x) = \lim_{n\to\infty} f_n(x)$. As in [12] it follows that $f(x)\in G(x,f(x))$, $x\in M$.

THEOREM 2 Let (X,d) be a metric space, A a nonempty, closed and convex subset of a Banach space $(Y,\|\ \|)$, $T:A\to X$ a continuous mapping such that $\overline{T(A)}$ is compact in X, $h:[0,\infty)\to[0,\infty)$ a nondecreasing mapping such that $h(t)<t$ $(t>0)$, and $G:\overline{T(A)}\times A\to CCB(A)$ is such that 1., 2. and 3. in Lemma 2. hold for $M=\overline{T(A)}$.

Then there exists $x_o\in A$ so that $x_o\in G(Tx_o,x_o)$.

Proof: From Lemma 2, as in Theorem 1, we conclude that mapping $Q:A\to A$, $Qx = f(Tx)$ ($x\in A$) has a fixed point $x_o\in Qx_o$.

Remark: If $h(u) = k\,u$ $(u>0)$ and $k\in(0,1)$ then for every $r>1$ such that $kr<1$ it follows that :

$$t_{n+1} = (1+kr+k^2r^2+\ldots+k^nr^n)\,t \quad ,n\in N$$

is bounded for every $t>0$. Hence, Theorem 2 is a generalization of Rzepecki's fixed point theorem from [21].

Theorems 1. and 2. can be generalized to paranormed spaces.

From Theorem D, using Example 1.3 from [17] we obtain the following result.

PROPOSITION Let X be a paracompact topological space, (Y,q) a complete paranormed space, $F:X\to CC(Y)$ a lower semicontinuous mapping and A a closed subset of X. If co $F(X)$ satisfies the Zima condition then every continuous selection $g:A\to Y$ of $F|A$ has an extension $f:X\to Y$ which is a continuous selection of F.

Proof: Let us define the mapping $H:X\to CC(Y)$ by :

$$H(x) = g(x)\ ,\text{for every } x\in A \text{ and } H(x) = F(x)$$

, for every $x\in X\setminus A$. Then [17] H is lower semicontinuous and co $H(X)\subset$ co $F(X)$ which implies that co $H(X)$ satisfies the Zima condition. Then from Theorem D it follows that there exists a continuous selection $f:X\to Y$ of the mapping H, which is obviously a continuous selection of F .

Now, we shall prove a generalization of Theorem 1 for paranormed spaces. In the next Theorem $C = C(\text{co } G(\overline{T(A)},A))$.

THEOREM 3 Let (X,d) be a metric space, A a nonempty, closed and convex subset of a complete paranormed space (Y,q) and satisfy the Zima condition, $T:A\to X$ a continuous mapping such that $\overline{T(A)}$ is compact in X , $G:\overline{T(A)}\times A\to A$ and the following conditions are satisfied :

(i)' The condition (i) in Lemma 1 .

(ii)' For every $x\in \overline{T(A)}$ and every $u,v\in A$:

$$D(G(x,u),G(x,v))\leqslant h(q(u-v))$$

where $h:[0,\infty)\to[0,\infty)$ is such that the mapping $\bar{h}$ defined by $\bar{h}(t) = C\, h(t)$, for every $t\geqslant 0$, satisfies $(A_1),(A_2),(A_4)$ and $\lim_{n\to\infty} \bar{h}^n(a) = 0$, for every $a>0$.

Then there exists $x_o\in A$ such that $x_o\in G(Tx_o,x_o)$.

Proof: Since the set A satisfies the Zima condition and A is a convex set it follows that the set co $G(\overline{T(A)},A)$, which is contained in the set A , satisfies the Zima condition also. Similarly as in Lemma 1 , we can prove the existence of the continuous mapping $f:\overline{T(A)}\to A$ such that $f(x)\in G(x,f(x))$, for every $x\in\overline{T(A)}$. Namely, since we can apply the Proposition we obtain on the right side of (6) $C(h(r(g_1,g_2)) + \frac{t}{2})$, which implies that $D_r(F(g_1),F(g_2))\leqslant C\, h(r(g_1,g_2))$, for every $g_1,g_2\in U$.

From Theorem B it follows the existence of the mapping f and from the Zima generalization of the Schauder fixed point theorem [27] we obtain that there exists $x_o\in A$ such that $x_o\in G(Tx_o,x_o)$.

Let us give an application of Theorem 1 on the existence of a solution of functional equation (1) .

First, we shall give some notations .

$I = [0,a]$ $(a>0)$, $f:I\times R^n\times R^n \to CCB(R^n)$, $m\in L^1(I,R^n) =$
$= \{x \mid x\in S(I,R^n)$, $\int_I |x(t)|_{R^n}\, dt<\infty \}$, $A = \{w \mid w\in L^1(I,R^n)$,

$|w(t)|\leqslant m(t)$ a.e. on I} , $T:A\to S(I,R^n)$ a compact mapping and $k:[0,a]\times R_+\to R_+$ so that $k(\cdot,s)\in L^1(I,R_+)$ $(s\in R_+)$.

THEOREM 4 Suppose that the mappings f and k satisfy the following conditions :

1. For every $(u,v)\in R^n\times R^n$, $t\mapsto f(t,u,v)$ is a measurable mapping and for every $t\in I$, $(u,v)\mapsto f(t,u,v)$ is a continuous

mapping.

2. For every $(t,u,v)\in I\times R^n\times R^n$, $D(f(t,u,v),\{\theta\})\leq m(t)$.
3. For every $(t,u,v_i)\in I\times R^n\times R^n$ $(i\in\{1,2\})$:
$$D(f(t,u,v_1),f(t,u,v_2))\leq k(t,|v_1-v_2|_{R^n}) .$$
4. For every $t\in[0,a]$, $k(t,\cdot)$ is a nondecreasing mapping such that the mapping $h:R_+\to R_+$, defined by $h(s)=\int_0^a k(t,s)dt$ satisfies all the conditions of Lemma 1 .

Then there exists a solution $g\in L^1(I,R^n)$ of the functional equation (1) .

Proof: Let $Y = L^1(I,R^n)$ and for every $x\in S(I,R^n)$ and $y\in A$
$$G(x,y)=\{u\,|\,u:I\to R^n, u \text{ is measurable }, u(t)\in f(t,x(t),\int_0^t y(u)du), t\in I\} .$$
Then [14] $G(x,y)\neq\emptyset$, for every $x\in S(I,R^n)$ and every $y\in A$ and $G(x,y)\in CCB(A)$. Further, the set A is closed and convex and $G:\overline{T(A)}\times A\to A$.

Let us prove that for every $x\in S(I,R^n)$ and $y_i\in A(i\in\{1,2\})$:

$$(7)\qquad D_{L^1(I,R^n)}(G(x,y_1),G(x,y_2))\leq h(|y_1-y_2|_{L^1(I,R^n)}) .$$

Let $w_1\in G(x,y_1)$. Then (see, for example [4], Lemma 23.2) there exists a measurable selector w_2 of the mapping $t\mapsto f(t,x(t),\int_0^t y_2(u)du)$ so that for every $t\in I$:

$$|w_1(t)-w_2(t)|_{R^n} = D(f(t,x(t),\int_0^t y_1(u)du),f(t,x(t),\int_0^t y_2(u)du)) .$$

From this we have that :
$$|w_1(t)-w_2(t)|_{R^n}\leq k(t,|\int_0^t[y_1(u)-y_2(u)]du|_{R^n})$$
$$\leq k(t,|y_1-y_2|_{L^1(I,R^n)})$$
and so $|w_1-w_2|_{L^1(I,R^n)}\leq\int_I k(t,|y_1-y_2|_{L^1(I,R^n)})dt=h(|y_1-y_2|_{L^1(I,R^n)})$

Similarly, it can be proved that for every $w_2'\in G(x,y_2)$ there exists $w_1'\in G(x,y_1)$ so that: $|w_1'-w_2'|_{L^1(I,R^n)}\leq h(|y_1-y_2|_{L^1(I,R^n)})$ which implies (7) . Using the same method we can prove that for every $y\in A$, the mapping $x\mapsto G(x,y)$ $(x\in S(I,R^n))$ is continuous .

Since T is a compact mapping, all the conditions of Theorem 1 are satisfied .

Hence, there exists $g \in A$ such that $g \in G(Tg,g)$ which means that g is a solution of functional equation (1). Functional equations (1) include differential inclusions [1].

Some interesting examples of compact integral operators in the space $S(I,R^n)$ are given in [13].

3. AN EXISTENCE RESULT FOR THE EQUATION $x(t) \in F(t,x)$

In this section we shall use the following notations: $(X,\| \ \|)$ is a uniformly convex Banach space; $I = [0,a]$ $(a>0)$ $C(I,X)$ is the space of all continuous functions from I into X, $R_+ = [0,\infty)$, $x_o \in C(I,X)$, $u_o \in C(I,R_+)$; $V(u_o) = \{ u \mid u \in C(I,R),$ $0 \leq u(t) \leq cu_o(t)$, for some $c>0$ and every $t \in I\}$; $D(x_o,u_o) = \{x \mid x \in C(I,X),\ \|x(t)-x_o(t)\| \leq cu_o(t)$, for some $c>0$ and every $t \in I\}$ and $F: I \times C(I,X) \to CCB(X)$.

Similarly as in [15] let us introduce the following conditions (i)-(iv):

(i) For each $x \in C(I,X)$ the mapping $F(\cdot,x): I \to CCB(X)$ is continuous.

(ii) $Q: V(u_o) \to V(u_o)$ is a nondecreasing operator such that:

$$D(F(t,x),F(t,y)) \leq Q(\|x-y\|)(t)$$

for every $x,y \in D(x_o,u_o)$ and every $t \in I$.

(iii) There exists a $p>0$ such that:

$$D(F(t,x_o),\{x_o(t)\}) \leq p\, u_o(t) \quad ,\text{for every } t \in I.$$

(iv) There exists a function $h: R_+ \to R_+$ which satisfies conditions (A_1)-(A_4) and:

$$Q(su_o)(t) \leq h(s)u_o(t) \ ,t \in I \ ,\text{for every } s>0.$$

THEOREM 5. <u>Let all the conditions</u> (i)-(iv) <u>be satisfied</u>. <u>Then there exists</u> $x \in C(I,X)$ <u>so that</u>:

$$x(t) \in F(t,x) \ ,t \in I.$$

Proof: For every $x \in C(I,X)$ let:

$$G(x) = \{f \mid f \in C(I,X), f(t) \in F(t,x), t \in I\}.$$

From the continuity of the mapping $t \mapsto F(t,x)$, for every $x \in C(I,X)$ since $F(t,x) \in CCB(X)$, it follows by Michael's selection theorem that $G(x) \neq \emptyset$, for every $x \in C(I,X)$. Let for every $x,y \in D(x_o,u_o)$:

$$d_1(x,y) = \sup_{t\in I_o} \frac{\|x(t)-y(t)\|}{u_o(t)}$$

where $I_o=\{t|t\in I \ ,u_o(t)\neq \emptyset\}$.

Then $(D(x_o,u_o),d_1)$ is a complete metric space . It is obvious that $G(x)$ is closed in $D(x_o,u_o)$,for every $x\in D(x_o,u_o)$.

Let $x\in D(x_o,u_o)$ and $f\in G(x)$. Since $t\mapsto F(t,x_o)$ is a continuous mapping from I into $CB(X)$ and $(X,\|\ \|)$ is a uniformly convex Banach space from Lemma 5.2 [2] it follows that there exists $g\in C(I,X)$ so that $g(t)\in F(t,x_o)$,for every $t\in I$ and :

$$\|f(t)-g(t)\| = \inf_{z\in F(t,x_o)} \|f(t)-z\| \ .$$

Then we have

$$\|f(t)-x_o(t)\|\leq\|f(t)-g(t)\|+\|g(t)-x_o(t)\|\leq D(F(t,x),F(t,x_o)) +$$
$$+\|g(t)-x_o(t)\|\leq Q(\|x-x_o\|)(t)+D(F(t,x_o),\{x_o(t)\})$$
$$\leq Q(cu_o)(t)+pu_o(t)\leq (h(c)+p)u_o(t).$$

which implies that :

$$d_1(f,x_o)\leq h(c)+p \quad ,f\in D(x_o,u_o) \ .$$

Hence $G(x)\in CB(D(x_o,u_o))$.

Let us prove that:

(8) $\bar{D}(G(x),G(y))\leq h(d_1(x,y))$,for every $x,y\in D(x_o,u_o)$

where $\bar{D}$ is the Hausdorff metric induced by the metric d_1 . In order to prove (8) we shall show that for every $f\in G(x)$ there exists $f_1\in G(y)$ so that $d_1(f,f_1)\leq h(d_1(x,y))$. If $f\in G(x)$ then

$$\|f(t)-f_1(t)\| = \inf_{z\in F(t,y)}\|f(t)-z\| =D(\{f(t)\},F(t,y))\leq$$

$\leq D(F(t,x),F(t,y))\leq Q(\|x-y\|)(t)$ for some $f_1\in G(y)$ [2] .

From $\|x(t)-y(t)\|\leq u_o(t)d_1(x,y)$ $(t\in I)$ it follows that

(9) $$\|f(t)-f_1(t)\|\leq h(d_1(x,y))u_o(t)\ ,t\in I$$

and (9) implies that:

(10) $$d_1(f,f_1) \leq h(d_1(x,y))$$

Similarly ,we can prove that for every $u_1\in G(y)$ there exists $u\in G(x)$ so that :

(11) $$d_1(u,u_1)\leq h(d_1(x,y))$$

Hence,(10) and (11) implies that (8) holds .

Applying Theorem B we conclude that there exists $x\in D(x_o,u_o)$ so that $x\in G(x)$ which means that $x(t)\in F(t,x)$,$t\in I$.

Using Theorem 5 ,similarly as in [15] ,some sufficient conditions for the existence of a solution of functional equations $x(t)\in F(t,x(r(t))$ and integro-functional equations

$$x(t)\in f(t,\int_0^{s(t)} g(t,u,x(u))du,x(r(t))) ,t\in I$$

can be given .Here,$r:I\to I$ and $s:I\to I$ are continuous mappings.

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REZIME

NEKE TEOREME O NEPOKRETNOJ TAČKI U BANAHOVIM PROSTORIMA I NJIHOVA PRIMENA

U ovom radu je dobijeno višeznačno uopštenje poznate teoreme Melvina o nepokretnoj tački [16].

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