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NATURAL EQUIVALENCE OF GRAPH RELATED BIFUNCTORS

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ABSTRACT

We introduce a method of investigating both topological spaces and graphs through a suitable notion of homotopy for certain classes of functions of graphs into hyperspaces of topological spaces. We show that this approach is naturally equivalent to the Demaria's method which is based on the notion of homotopy for certain classes of functions of topological spaces into graphs.

1. INTRODUCTION.

There has been a considerable research by D.C.Demaria, M.Burzio, G.M.Gianella, C.Guido, B.Casciaro, F.Cammaroto, L.Carini, P.M.Gandini and O.M.Amici on the theory of regular homotopy of regular functions into graphs. This theory can be vaguely described as follows.

One first introduces a class of regular functions from a topological space X into a graph G as functions $f:X \to G$ for which the preimages of nonadjacent vertices are topologically separated. Regular functions $f, g:X \to G$ are homotopic provided there is a regular function $H:X \times [0,1] \to G$ such that

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$H(x,0)=f(x)$ and $H(x,1)=g(x)$ for every $x \in X$. The regular homotopy theory is simply the study of the bifunctor [5] r which associates to a topological space X and a graph G the set $r[X:G]$ of all equivalence classes of regular functions of X into G with respect to the equivalence relation of homotopy. Of course the bifunctior r determines two functors: r_G (when G is regarded fixed) and r^X (when X is regarded fixed) on the category of topological spaces and on the category of graphs, respectively.

In this paper, we shall reverse the above procedure and consider multivalued functions of a graph G into a topological space X instead of functions of X into G. The multivalued function $\varphi: G \to X$ is regular provided the images of nonadjacent vertices are topologically separated. Two regular multivalued functions $\psi, \varphi: G \to X$ are homotopic provided there is a regular multivalued function $\Phi: G \to X \times [0,1]$ that connects φ and ψ. Thus, we can define a bifunctor c which associates to a topological space X and a graph G the set $c[G:X]$ of all equivalence classes of regular multivalued functions of G into X with respect to the equivalence relation of homotopy.

Our main result shows that the bifunctors r and c are naturally equivalent. It follows that the two approaches are only formally different. However, while Demaria's regular functions could be regarded as a realization (analogous to the intersection graphs [4]) of a graph in terms of covers of a topological space into pairwise disjoint subsets, in our approach this last condition has been eliminated. This has some obvious advantages. For example, our point of view makes the whole subject a part of the hyperspace theory of topological spaces so that we can use the machinery of hyperspaces, multivalued functions, and their selections and easily apply the method to other structures.

We also show that the similar statements hold for other types of regular functions and multivalued functions (completely regular and 0-regular and 0*-regular for directed graphs), and for the case of pairs and more generally of n-tuples.

2. R-FUNCTIONS AND R-HOMOTOPY

In this section we shall recall Demaria's [1] notion of a regular function (or an r-function) of a space into a graph and the associated equivalence relation of homotopy for r-functions.

Let G be a graph. A pair $\{v,w\}$ of elements of G will be called a j-pair provided either $v=w$ or v and w are end-points of an edge in G. The pairs of vertices which are not j-pairs are called d-pairs.

A function $f:X \to G$ of a topological space X into a graph G is an r-function provided $\overline{f^{-1}(v)} \cap f^{-1}(w) = f^{-1}(v) \cap \overline{f^{-1}(w)} = \emptyset$ for every d-pair $(v,w) \in G \times G$. Two r-functions $f,g:X \to G$ are r-homotopic (in notation $f \overset{r}{\simeq} g$) provided there exists an r-function $F:X \times I \to G$ (called an r-homotopy) such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for every $x \in X$. The relation of r-homotopy is an equivalence relation on the set $r(X;G)$ of all r-functions of X into G. Let $[f]$ denote the equivalence class of $f \in r(X;G)$ and let $r[X;G]$ denote the quotient set. Moreover, we can consider the association $(X,G) \to r[X;G]$ as a bifunctor [5, p. 39] $r:\tau^{o} \times \mathcal{G} \to \mathcal{S}$, where τ^{o} is the opposite to the category τ of topological spaces and continuous functions, $\mathcal{G}$ is the category of graphs and j-functions (i.e., finite-to-one functions which map j-pairs into j-pairs), and $\mathcal{S}$ is the category of sets and functions. On morphisms r is defined as follows. Let $(f,m) \in \mathrm{Mor}_{\tau^{o}\times\mathcal{G}}((X,G),(Y,H))$. Then $r(f,m):r[X;G] \to r[Y;H]$ is given by $r(f,m)([g]) = [m \circ g \circ f]$ for $g \in r(X;G)$.

For pairs the above notions are defined as follows. In an analogous way it is possible to consider those concepts also for arbitrary n-tuples.

Let X be a topological space and let Y be a subspace of X. Let G be a graph and let H be a subgraph of G. Call a function $f:(X,Y) \to (G,H)$ between pairs an r-<u>function</u> provided $f:X \to G$ and $f|_Y:Y \to H$ are both r-functions. Two r-functions $f, g:(X,Y) \to (G,H)$ are r-<u>homotopic</u> provided there is an r-function $F:(X \times I, Y \times I) \to (G,H)$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for every $x \in X$. The relation of r-homotopy is an equivalence relation on the set $r(X,Y;\, G,H)$ of all r-functions from (X,Y) into (G,H). The equivalence class of an $f \in r(X,Y;\, G,H)$ is denoted by $[f]$ while $r[X,Y;\, G,H]$ denotes the set of all equivalence classes.

Just like in the absolute case, we can consider the association $((X,Y),(G,H)) \to r[X,Y;\, G,H]$ as a bifunctor $r_2:\tau_2^o \times \mathcal{G}_2 \to \mathcal{S}$, where τ_2^o is the opposite to the category τ_2 of pairs (X,Y) consisting of a topological space X and its subspace Y and continuous functions of such pairs and $\mathcal{G}_2$ is the category of pairs (G,H) consisting of a graph G and a subgraph H of G and j-functions of such pairs.

When (X,Y) is a pair $(I^n, \dot{I}^n)$ consisting of the n-cube and its boundary $\dot{I}^n$ and (G,H) is a pair (G,v) consisting of the graph G and a vertex $v \in G$, the set $r_n(G,v) = r[I^n,\dot{I}^n);\, G,v]$ becomes a group under the binary operation o defined as follows. For $[f], [g] \in r_n(G,v)$, put $[f]o[g] = [h]$ where $h:(I^n,\dot{I}^n) \to (G,v)$ is given by the formula

$$h(t_1,\dots,t_n) = \begin{cases} f(3t_1,t_2,\dots,t_n), & t_1 \in [0,1/3] \\ f(1,t_2,\dots,t_n), & t_1 \in [1/3,2/3] \\ g(3t_1-2,t_2,\dots,t_n), & t_1 \in [2/3,1] \end{cases}$$

for every $(t_1,\dots,t_n) \in I^n$.

3. C-FUNCTIONS AND C-HOMOTOPY

This section introduces a class of multivalued functions (called c-functions) from a graph into a space and the associated equivalence relation of homotopy for c-functions.

Let a^*X denote the hyperspace of all subsets of a topological space X and let $p_X : X \times I \to X$ be the natural projection.

DEFINITION 3.1. *A function* $\varphi : G \to a^*X$ *of a graph* G *into* a^*X *is a c-function provided*

(i) $X = \cup \{\varphi(v) \mid v \in G\}$,

and

(ii) $\overline{\varphi(v)} \cap \varphi(w) = \varphi(v) \cap \overline{\varphi(w)} = \emptyset$

for every d-*pair* $(v,w) \in G \times G$.

Two c-functions $\varphi, \psi : G \to a^*X$ are c-<u>homotopic</u> (in notation, $\varphi \overset{c}{\simeq} \psi$) provided there exists a c-function $\Phi : G \to a^*(X \times I)$ (called a c-homotopy) such that $\varphi(v) = p_X(\Phi(v) \cap (X \times \{0\}))$ and $\psi(v) = p_X(\Phi(v) \cap (X \times \{1\}))$ for every $v \in G$.

THEOREM 3.2. *The relation* $\overset{c}{\simeq}$ *is an equivalence relation on the set* $c(G; X)$ *of all c-functions of* G *into* X.

P r o o f. a) (Reflexive) For $\varphi \in c(G;X)$, the function $\Phi : G \to a^*(X \times I)$ defined by $\Phi(v) = \varphi(v) \times I$ for each $v \in G$ is a c-function and $p_X(\Phi(v) \cap (X \times \{0\})) = p_X(\Phi(v) \cap (X \times \{1\})) = \varphi(v)$ for every $v \in G$. Hence, $\varphi \overset{c}{\simeq} \varphi$.

b) (Symmetric) Suppose $\varphi, \psi \in c(G; X)$ and $\varphi \overset{c}{\simeq} \psi$. Let $\Phi : G \to a^*(X \times I)$ be a c-homotopy between φ and ψ. Let $r : X \times I \to X \times I$ be defined by $r(x,t) = (x, 1-t)$ for each $(x,t) \in X \times I$. Define $\psi : G \to a^*(X \times I)$ by $\psi(v) = r(\Phi(v))$ for every $v \in G$. One can easily check that ψ is a c-homotopy between ψ and φ. Hence $\psi \overset{c}{\simeq} \varphi$.

c) (Transitive) Suppose $\varphi, \psi, \chi \in c(G;\ X)$ and $\varphi \overset{c}{\simeq} \psi$ and $\psi \overset{c}{\simeq} \chi$. Let Φ and ψ be c-homotopies between φ and ψ and between ψ and χ, respectively. Define maps $m: X \times I \to X \times [0,1/2]$ and $n: X \times I \to X \times [1/2,1]$ by $m(x,t) = (x,t/2)$ and $n(x,t) = (x,(1/2) + (t/2))$ for each $(x,t) \in X \times I$. Now, define $\Sigma: G \to a^*(X \times I)$ by $\Sigma(v) = m(\Phi(v)) \cup n(\psi(v))$, for each $v \in G$. Clearly, Σ satisfies the condition (3.1)(i) and $\varphi(v) = p_X(\Sigma(v) \cap (X \times \{0\})$ and $\chi(v) = p_X(\Sigma(v) \cap (X \times \{1\}))$ for every $v \in G$. Hence, Σ is a c-homotopy between φ and χ, and therefore $\varphi \overset{c}{\simeq} \chi$, provided we prove that Σ also satisfies the condition (3.1) (ii).

Let $(v,w) \in G \times G$ be a d-pair. Then $\overline{\Sigma(v)} \cap \Sigma(w) = [\overline{m(\Phi(v))} \cup \overline{n(\psi(v))}] \cap [m(\Phi(w)) \cup n(\psi(w))] = A \cup B \cup C \cup D$, where $A = \overline{m(\Phi(v))} \cap m(\Phi(w))$, $B = \overline{m(\Phi(v))} \cap n(\psi(w))$, $C = \overline{n(\psi(v))} \cap m(\Phi(w))$, and $D = \overline{n(\psi(v))} \cap n(\psi(w))$. Since Φ and ψ are c-functions and m and n are homeomorphisms, $A = D = \emptyset$. We claim that $B = \emptyset$ (and analogously, that $C = \emptyset$).

Indeed, suppose $B \neq \emptyset$. Then there is an $x \in X$ such that $(x,1/2) \in B$. It follows that $(x,1) \in \overline{\Phi(v)}$ and $(x,0) \in \psi(w)$. Hence, $x \in p_X(\psi(w) \cap (X \times \{0\})) = \psi(w) = p_X(\Phi(w) \cap (X \times \{1\}))$ and $(x,1) \in \Phi(w)$. However, this is impossible because Φ is a c-function so that $\overline{\Phi(v)} \cap \Phi(w) = \emptyset$

For a c-function $\varphi: G \to a^*X$, let $[\varphi]$ denote the equivalence class of φ with respect to the relation $\overset{c}{\simeq}$ (called a c-homotopy class of φ) and let $c[G;X]$ denote the quotient set $c(G;X) / \overset{c}{\simeq}$.

Moreover, we can consider the association $(X,G) \mapsto c[G;\ X]$ as a bifunctor $c: \tau^{\circ} \times G \to S$ provided we define c on morphisms as follows. Let $(f,m) \in \mathrm{Mor}_{\tau^{\circ} \mathcal{G}}((X,G),(Y,H))$. Then $c(f,m): c[G;X] \to c[H;Y]$ is given by $c(f,m)([\varphi]) = [\psi]$, where $\varphi \in c(G;X)$ and $\psi(w) = \cup_{v \in m^{-1}(w)} f^{-1}(\varphi(v))$ for $w \in H$.

The above notions are defined for pairs as follows.

In an analogous fashion they can be also introduced for n-tuples.

Let X be a topological space and let Y be a subspace of X. Let G be a graph and let H be a subgraph of G. A c-function $\varphi: G \to a^*X$ is a c-function from (G,H) into (X,Y) (in notation, $\varphi:(G,H) \to (X,Y)$) provided $\varphi(v) \subset Y$ for every $v \in H$. Two c-functions $\varphi,\psi:(G,H) \to (X,Y)$ are c-homotopic provided there is a c-function $\Phi:(G,H) \to (X \times I, Y \times I)$ such that $\varphi(v) = p_X(\Phi(v) \cap (X \times \{0\}))$ and $\psi(v) = p_X(\Phi(v) \cap (X \times \{1\}))$ for every $v \in G$. One can prove that the relation of c-homotopy is an equivalence relation on the set $c(G,H;X,Y)$ of all c-functions from (G,H) into (X,Y). The equivalence class of a $\varphi \in c(G,H;X,Y)$ is denoted by $[\varphi]$ while $c[G,H;X,Y]$ denotes the set of all equivalence classes. Just like in the absolute case, we can consider the association $((X,Y),(G,H)) \mapsto c[G,H,X,Y]$ as a bifunctor $c_2: \tau_2^o \times \mathcal{G}_2 \to \mathcal{S}$.

When (X,Y) is the pair $(I^n, \dot{I}^n)$ and (G,H) is the pair (G,v), the set $c_n(G,v) = c[G,v;I^n,\dot{I}^n)$ becomes a group under the binary operation o defined as follows. Let $m,q: I^n \to I^n$ be defined by $m(t_1,\ldots,t_n) = (t_1/2,t_2,\ldots,t_n)$ and $q(t_1,\ldots,t_n) = ((1+t_1)/2,t_2,\ldots,t_n)$. Now, for $[\varphi]$, $[\psi] \in c_n(G,v)$ put $[\varphi]o[\psi] = [\chi]$ where the c-function $\chi:(G,v) \to (I^n,\dot{I}^n)$ is given by the formula $\chi(w) = m(\varphi(w)) \cup q(\psi(w))$ for every $w \in G$.

4. R-HOMOTOPY VERSUS C-HOMOTOPY

The main results in this section (Theorem 4.1)) shows that the bifunctors c and r are naturally equivalent. It follows that r-homotopy and c-homotopy are two equivalent approaches in the investigation of properties of topological spaces using graphs and in the investigation of properties of graphs using topological spaces.

THEOREM 4.1. *There is a natural equivalence* θ *between bifunctors* c *and* r.

P r o o f. In order to prove the theorem, for each topological space X and each graph G, we shall define a bijection $\theta = \theta(X,G): c[G;X] \to r[X;G]$ such that for every morphism $(f,m) \in \mathrm{Mor}_{T^o \times G}((X,G),(Y,H))$ the diagram

$$\begin{array}{ccccc} & c[G;X] & \xrightarrow{\ c(f,m)\ } & c[H;Y] & \\ \theta(X,G) & \downarrow & & \downarrow & \theta(Y,H) \\ & r[X;G] & \xrightarrow[\ r(f,m)\ ]{} & r[Y;H] & \end{array}$$

commutes. This will be accomplished in the following six lemmas.

DEFINITION OF $\theta(X,G)$: Let $[\varphi] \in c[G;X]$. For each $x \in X$ let $\varphi^{-1}(x) = \{v \in G \mid x \in \varphi(v)\}$. We can consider φ^{-1} as a function of X into the hyperspace aG of all non-empty subsets of G.

LEMMA 4.2. *Every selection* $f: X \to G$ *of the function* φ^{-1} *is an r-function.*

P r o o f. Recall that f is a selection for φ^{-1} provided $f(x) \in \varphi^{-1}(x)$ for every $x \in X$. Hence $x \in \varphi(f(x))$ for every $x \in X$.

Let $(v,w) \in G \times G$ be a d-pair. Since φ is a c-function, $\overline{\varphi(v)} \cap \varphi(w) = \emptyset$. But, $f^{-1}(v) \subset \varphi(v)$ and $f^{-1}(w) \subset \varphi(w)$ so that $\overline{f^{-1}(v)} \cap f^{-1}(w) = \emptyset$. Hence, f is an r-function.

LEMMA 4.3. *Every two selections* $f_1, f_2: X \to G$ *of the function* φ^{-1} *are r-homotopic.*

P r o o f. Define a function $F: X \times I \to G$ as follows.

$$F(x,t) = \begin{cases} f_1(x), & (x,t) \in X\times[0,1/2) \\ f_2(x), & (x,t) \in X\times[1/2,1] \end{cases}$$

We claim that F is an r-homotopy between f_1 and f_2.

In order to verify this claim it clearly suffices to show that F satisfies the condition (3.1)(ii).

Consider a d-pair $(v,w) \in G\times G$. Then $\overline{F^{-1}(v)} \cap F^{-1}(w) = A\cup B\cup C\cup D$, where $A=P\cap R$, $B=P\cap S$, $C=Q\cap R$, $D=Q\cap S$, $P=f_1^{-1}(v)\times[0,1/2]$, $Q=\overline{f_2^{-1}(v)}\times[1/2,1]$, $R=f_1^{-1}(w)\times[0,1/2]$, and $S=f_2^{-1}(w)\times[1/2,1]$. Since f_1 and f_2 are r-functions, $A=D=\emptyset$. We claim that $B=\emptyset$ (and, analogously, that $C=\emptyset$).

Indeed, if $(x,1/2)\in B$, then $x\in\overline{f^{-1}(v)}\cap f^{-1}(w)\subset\overline{\varphi(v)}\cap\varphi(w)$. However, the intersection $\overline{\varphi(v)}\cap\varphi(w)$ is empty because φ is a c-function and (v,w) is a d-pair. Hence, $B=\emptyset$.

LEMMA 4.4. *Let* $\varphi,\psi: G\to a^*X$ *be c-functions and let* $f:X\to G$ *and* $g:X\to G$ *be selections of* $\varphi^{-1}:X\to aG$ *and* $\psi^{-1}:X\to aG$, *respectively. If* $\varphi\overset{c}{\simeq}\psi$, *then* $f\overset{r}{\simeq}g$.

P r o o f. Let $\Phi: G\to a^*(X\times I)$ be a c-homotopy between φ and ψ. Let $F:X\times I\to G$ be a selection for the function $\Phi^{-1}:X\times I\to aG$. We shall prove that $F_0:X\to G$ defined by $F_0(x) = F(x,0)$ (for $x\in X$) is a selection for φ^{-1}. Then it follows from the Lemma 4.3. that $f\overset{r}{\simeq}F_0$. Since one can similarly show that $g\overset{r}{\simeq}F_1$, it follows that $f\overset{r}{\simeq}g$.

Let $x\in X$. Since F is a selection for Φ^{-1}, $F(x,0)\in \Phi^{-1}(x,0)$. Hence, $(x,0)\in\Phi(F(x,0))$ and $x\in p_X(\Phi(F(x,0))\cap(X\times\{0\})) = \varphi(F(x,0))$. Finally, we get $F_0(x)=F(x,0)\in\varphi^{-1}(x)$, i.e. that F_0 is a selection for φ^{-1}.

It follows from the Lemmas 4.2.-4.4. that the function $\theta = \theta(X,G): c[G;X] \to r[X;G]$ which associates to a class $[\varphi] \in c[G;X]$ the r-homotopy class $[f]$ of any selection f of the function φ^{-1} is well-defined.

LEMMA 4.5. *The function* θ *is one-to-one.*

P r o o f. Consider $[\varphi], [\psi] \in c[G;X]$. Suppose $\theta([\varphi]) = \theta([\psi])$. We shall prove that $\varphi \overset{c}{\simeq} \psi$.

Let $f: X \to G$ be a selection of the function φ^{-1} and let $g: X \to G$ be a selection of the function ψ^{-1}. Let $H: X \times I \to G$ be an r-homotopy between f and g. Define functions $\chi: G \to a^*(X \times \times I)$, $\alpha: G \to a^*X$, and $\beta: G \to a^*X$ by $\chi(v) = H^{-1}(v)$, $\alpha(v) = p_X(\chi(v) \cap \cap (X \times \{0\}))$, and $\beta(v) = p_X(\chi(v) \cap (X \times \{1\}))$ for every $v \in G$, respectively.

In order to show that φ and ψ are c-homotopic we shall prove that $\alpha \overset{c}{\simeq} \psi$ and $\beta \overset{c}{\simeq} \varphi$. Since χ is a c-homotopy between α and β, this would imply $\varphi \overset{c}{\simeq} \psi$.

Let a homeomorphism $m: X \times I \to X \times [0,1/3]$ be defined by $m(x,t) = (x,t/3)$ for each $(x,t) \in X \times I$. Next, we define a function $k: G \dashv a^*(X \times I)$ by $k(v) = m(\chi(v)) \cup (g^{-1}(v) \times [1/3,2/3] \cup (\psi(v) \times [2/3,1])$. We claim that k is a c-homotopy between α and ψ.

Indeed, let us check that the function k satisfies the condition (3.1)(ii). The other required properties of k are easy to verify.

Let $(v,w) \in G \times G$ be a d-pair. Then $\overline{k(v)} \cap k(w) = A \cup B \cup C \cup D \cup E \cup F \cup J$, where $A = P \cap P'$, $B = P \cap Q'$, $C = Q \cap P'$, $D = Q \cap Q'$, $E = Q \cap R'$, $F = R \cap Q'$, $J = R \cap R'$, $P = \overline{\psi(v)} \times [2/3,1]$, $Q = g^{-1}(v) \times [1/3,2/3]$, $R = \overline{m(\chi(v))}$, $P' = \psi(w) \times [2/3,1]$, $Q' = g^{-1}(w) \times [1/3,2/3]$ and $R' = m(\chi(w))$. Since ψ, g^{-1}, and χ are c-functions, $A = D = J = \emptyset$. We shall now prove that $B = \emptyset$ and $E = \emptyset$. One can analogously prove that $C = \emptyset$ and $F = \emptyset$. It follows that $\overline{k(v)} \cap k(w) = \emptyset$. Since a similar argument shows that $k(v) \cap \overline{k(w)} = \emptyset$, we get $\alpha \overset{c}{\simeq} \psi$. In an analogous fashion one can prove that $\beta \overset{c}{\simeq} \varphi$ and thus conclude the proof.

Suppose that there is an $x \in X$ such that $(x,2/3) \in C$. Then $x \in g^{-1}(v)$ and $x \in \psi(w)$. But, $g^{-1}(v) \subset \psi(v)$ so that $x \in \in \overline{\psi(v)} \cap \psi(w)$. This is impossible because ψ is a c-function.

Finally, suppose that there is an $x \in X$ such that $(x,1/3) \in E$. Then $x \in \overline{g^{-1}(v)}$ and $(x,1) \in H^{-1}(w)$. Hence, $H(x,1) = g(x) = w$ so that $x \in \overline{g^{-1}(v)} \cap g^{-1}(w)$. However, this is impossible because g^{-1} is a c-function.

LEMMA 4.6. *The function θ is onto.*

P r o o f. Let $[h] \in r[X;G]$. Put $\varphi = h^{-1}: G \to a^*X$. Since $\varphi^{-1}(x) = \{h(x)\}$ for every $x \in X$, we see that h is a selection of φ^{-1}. This implies that $\theta([\varphi]) = [h]$.

LEMMA 4.7. *For every morphism* $(f,m) \in \mathrm{Mor}_{\tau^{\circ}\times G}((X,G),(Y,H))$ *the equality* $\theta(Y,H) \circ c(f,m) = r(f,m) \circ \theta(X,G)$ *holds.*

P r o o f. Let $\psi \in c(G;X)$. Define $\chi \in c(H;Y)$ by $\chi(w) = \bigcup_{v \in m^{-1}(w)} f^{-1}(\psi(v))$ for $w \in H$. Let $g: Y \to G$ be a selection of $\psi^{-1}: Y \to aG$. It suffices to prove that $m \circ g \circ f$ is a selection of $\chi^{-1}: X \to aG$. Let $y \in Y$. Since g is a selection of ψ^{-1}, we have $f(y) \in \psi(g(f(y)))$ and $y \in f^{-1}[\psi(g(f(y)))]$. Hence, $y \in \in \chi[(m_o g_o f)]$. In other words, $(m_o g_o f)(y) \in \chi^{-1}(y)$ for every $y \in Y$.

REMARK 4.8. Only minor changes in the above proof are needed in order to prove the version of the Theorem 4.1 for pairs and n-tuples.

5. S-FUNCTIONS AND E-FUNCTIONS

Demaria [1] has also introduced the notion of a <u>strongly regular function</u> (or an s-<u>function</u>) as a function $f: X \to G$ of a topological space X into a graph G which satisfies the condition $\overline{f^{-1}(v)} \cap \overline{f^{-1}(w)} = \emptyset$ for every d-pair $(v,w) \in G \times G$.

By replacing throughout the section 2 the letter "r" with the letter "s" we can define the notion of s-homotopy and the bifunctor $s:\tau^{o}\times G\to S$. This could also be done for pairs and more generally for n-tuples. Moreover, all statements about r-functions and r-homotopy hold also for s-functions and s-homotopy. In particular, as the following lemma shows, the relation of s-homotopy is an equivalence relation contrary to the claim in [2,p. 142]

LEMMA 5.1. *The relation* $\overset{s}{\simeq}$ *is an equivalence relation on the set* $s(X;G)$ *of all* s-*functions of* X *into* G.

P r o o f. We leave to the reader to check that $\overset{s}{\simeq}$ is reflexive and symmetric. In order to prove that it is transitive, suppose $f,g,h\in s(X;G)$, $f\overset{s}{\simeq}g$, and $g\overset{s}{\simeq}h$. Let F and G be s-homotopies between f and g and between g and h, respectively. Define a function $H:X\times I\to G$ by

$$H(x,t) = \begin{cases} F(x,3t), & (x,t)\in X\times[0,1/3] \\ g(x), & (x,t)\in X\times[1/3,2/3] \\ G(x,3t-2), & (x,t)\in X\times[2/3,1]. \end{cases}$$

Then H is an s-homotopy between f and h.

In analogy with the section 3, we shall say that a function $\varphi:G\to a^{*}X$ of a graph G into the hyperspace $a^{*}X$ of all subsets of a topological space X is an e-function provided $X=\cup\{\varphi(v)\mid v\in G\}$ and $\overline{\varphi(v)}\cap\overline{\varphi(w)}=\emptyset$ for every d-pair $(v,w)\in G\times G$. The replacement of the letter "c" with the letter "e" in the section 3 gives us the notion of e-homotopy and the bifunctor $e:\tau^{o}\times G\to S$. Of course, this also applies to pairs and n-tuples.Moreover, only routine changes in the section 4 are needed to get the proof of the following result.

THEOREM 5.2. *The bifunctors* e *and* s *are naturally equivalent.*

Related to the bifunctor e is the bifunctor b defined as follows. Let c^*X denote the hyperspace of all closed subsets of a topological space X. A function $\varphi: G \to c^*X$ of a graph G into c^*X is a b-function provided $X = \cup\{\varphi(v) \mid v \in G\}$ and $\varphi(v) \cap \varphi(w) = \emptyset$ for every d-pair $(v,w) \in G \times G$. Two b-functions $\varphi, \psi: G \to c^*X$ are b-homotopic (in notation, $\varphi \overset{b}{\simeq} \psi$) provided there is a b-function $\Phi: G \to c^*(X \times I)$ such that $\varphi(v) = p_X(\Phi(v) \cap (X \times \{0\}))$ and $\psi(v) = p_X(\Phi(v) \cap (X \times \{1\}))$ for every $v \in G$. It is clear now how we can define the bifunctor b and that there is a "b-version" of all of the section 3.

THEOREM 5.3. *The bifunctors* e *and* b *are naturally equivalent.*

P r o o f. For a $\varphi \in e(G;X)$, define $\varphi^* \in b(G;X)$ by $\varphi^*(v) = \overline{\varphi(v)}$. Then φ^* is a b-function and the formula $\Phi(v) = [\varphi(v) \times [0,1)] \cup [\varphi^*(v) \times \{1\}]$ for $v \in G$ defines an e-homotopy between φ and φ^*. It is now routine to show that $\tau = \tau(X;G): e[G;X] \to b[G;X]$ given by $\tau([\varphi]) = [\varphi^*]$ defines a natural equivalence between e and b.

The following corollary of the theorems 5.3 and 5.2 improves the main result in [3].

COROLLARY. *The bifunctors* b *and* e *are naturally equivalent.*

REMARK 5.5. Just like in the section 4, all of the results in this section hold also for pairs and n-tuples.

6. DIGRAPHS

In the case when G is a directed graph (or a digraph), Demaria considered o-regular and o*-regular functions of a space X into G. In order to define these classes of functions, we shall call a pair $(v,w) \in G \times G$ a j-pair provided either

$v = w$ or (v,w) is an edge of G. An ordered pair of elements of G which is not a j-pair is called a d-pair.

A function $f:X \to G$ is an or-function (o*r-function) provided $f^{-1}(v) \cap f^{-1}(w) = \emptyset$ $(f^{-1}(v) \cap f^{-1}(w) = \emptyset)$ for every d-pair $(v,w) \in G \times G$. Of course, it is now possible to introduce or-homotopy and o*r-homotopy and bifunctors or and o*r. Also, we can define oc-functions and o*c-functions, corresponding homotopies, and bifunctors oc and o*c. The method of proof in the section 4 with minor alternations allows one to prove that oc is naturally equivalent to or and that o*c is naturally equivalent to o*r. A similar statement is also true for pairs and for n-tuples.

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REZIME

PRIRODNA EKVIVALENCIJA GRAFU ASOCIRANIH BIFUNKTORA

Uveden je metod za ispitivanje topoloških prostora i grafova pomoću podesnog pojma homotopije za neke klase presli-

kavanja grafova u hiperprostore topoloških prostora. Pokazano je da je ovaj pristup prirodno ekvivalentan metodi Demaria koji je zasnovan na pojmu homotopije za neke klase preslikavanja topoloških prostora u grafove.