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COINCIDENCE POINTS FOR SET-VALUED MAPPINGS IN CONVEX METRIC SPACES

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ABSTRACT

In this paper a generalization of the well-known fixed point theorem of Assad and Kirk for multivalued mappings in convex metric spaces is given. The multivalued version of the Palais-Smale condition is introduced and applied in the proof of Theorem 2., which contains an existence result on coincidence points for set-valued mappings in metric spaces with a convex structure.

1. INTRODUCTION

Since Assad and Kirk published their paper [1] many authors proved fixed point theorems or theorems on coincidence points in metric spaces with a convex structure or in convex metric spaces [2], [5], [6], [7], [8], [9], [10], [11], [12].

Let us recall that a metric space (M,d) is convex if for each $x,y \in M$ with $x \neq y$ there exists $z \in M$, $x \neq z \neq y$ such that

$$d(x,z) + d(z,y) = d(x,y).$$

In [1] the following result is obtained.

THEOREM A. *Let* (M,d) *be a complete convex metric space,* K *a nonempty closed subset of* $M, f: K \to CB(M)$ *(the family of all nonempty closed and bounded subsets of* M *) so that* $f(\partial K) \subseteq K$ *and* f *is a contraction mapping (in respect to the Hausdorff metric* H *). Then there exists*

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$z \in K$ *such that* $z \in fz$.

Itoh [7], Khan [8], Baskaran and Subrahmanyam [2] and Hadžić [6] (for singlevalued f) obtained generalizations of Theorem A. Theorem 1, which will be proved in this paper, is a theorem on the coincidence point for mappings $f:K \to CB(M)$ $S:K \to M$ and $T:K \to M$. If $S = T = Id|_K$ from Theorem 1 it follows Theorem A. A point $y \in K$ is a coincidence point for f, S and T if

$$\{Ty, Sy\} \subseteq fy .$$

We shall need for the next text some notions and notations. If X is a metric space, we shall denote by 2^X the family of all nonempty subsets of X. Let (X,d) be a metric space, $f:K \to CB(X)$, $S:K \to X$ and $\emptyset \neq K \subset X$. The pair (f,S) is said to be weakly commutative if and only if for every $y \in K$ and $z \in K$ such that $y \in fz$ and $Sz \in K$:

$$d(Sy, fSz) \leqslant d(fz, Sz) .$$

For singlevalued mappings the notion of the weak commutativity is introduced by Sessa in [11]. There are examples of mappings which are weakly commutative but not commutative. If f and S are such that for $Sz \in K$ and $fz \subset K$, $fSz = Sfz$ then the pair (f,S) is obviously weakly commutative since for $y \in fz \cap K$ and $Sz \in K$:

$$d(Sy, fSz) = \inf_{u \in fSz} d(Sy, u)$$

and there exists $v \in fSz$ such that $Sy = v$. This implies that

$$\inf_{u \in fSz} d(Sy,u) \leqslant d(Sy,v) = d(v,v) = 0 \leqslant d(fz,Sz)$$

In [12] Takahashi introduced the notion of the convexity in metric spaces. Let (X,d) be a metric space and $W: X \times X \times [0,1] \to X$. The mapping W is said to be a convex structure if for every $(x,y,\lambda) \in X \times X \times [0,1]$:

$$d\big(u, W(x,y,\lambda)\big) \leqslant \lambda d(u,x) + (1-\lambda) d(u,y)$$

for every $u \in X$. A metric space with a convex structure belongs to the class of convex metric spaces. There are metric spaces with a convex structure which can not be imbedded in any Banach space [12].

A metric space (X,d) with a convex structure W satisfies condition II if for all $(x,y,z,\lambda)\in X^3\times[0,1]$

$$d\big(W(x,z,\lambda),W(y,z,\lambda)\big) \leqslant \lambda d(x,y) \quad [10].$$

If (X,d) is a metric space with a convex structure $W, x_o\in X$ and $S:X\to X$ we say that the mapping S is (W,x_o)-convex if and only if for every $z\in X$ and every $\lambda\in(0,1)$:

$$W(Sz,x_o,\lambda) = SW(z,x_o,\lambda) \ .$$

By α we shall denote the Kuratowski measure of noncompactness.

2. A GENERALIZATION OF THE FIXED POINT THEOREM OF ASSAD AND KIRK

THEOREM 1. *Let* (M,d) *be a complete convex metric space,* K *a nonempty closed subset of* M, $S,T:K\to M$ *continuous mappings,* $f:K\to CB(M)$ *H-continuous mapping,* $\partial K\subseteq SK\cap TK$, $fK\cap K\subseteq SK\cap TK$, (f,S) *and* (f,T) *weakly commutative pairs and the following implications hold:*

$$Tx\in\partial K \Rightarrow fx\in K; \quad Sx\in\partial K \Rightarrow fx\in K.$$

If there exists $q\in(0,1)$ *so that:*

$$H(fx,fy)\leqslant qd(Sx,Ty)\ , \ \textit{for every}\ \ x,y\in K,$$

then there exists $z\in K$ *so that:*

$$\{Tz,Sz\}\cap fz\neq\emptyset\ .$$

If $S,T:M\to M$ *are continuous and:*

(i) $y\in fx,\ Tx\in K \Rightarrow d(Ty,fTx)\leqslant d(fx,Tx)$

(ii) $y\in fx,\ Sx\in K \Rightarrow d(Sy,fSx)\leqslant d(fx,Sx)$

then there exists $z\in K$ *so that:*

$$Tz\in fz \ \textit{and}\ Sz\in fz\ .$$

PROOF : Let $x\in\partial K$. Since $\partial K\subseteq TK$ it follows that there exists $p_o\in K$ such that $x=Tp_o$.

From $Tp_o\in\partial K$, using the implication: $Tx\in\partial K\Rightarrow fx\in K$, we conclude that $fp_o\in K\cap fK\subseteq SK$. Let $p_1\in K$ be such that $Sp_1 = p_1'\in\in fp_o\subseteq K$. Since $p_1'\in fp_o$ there exists $p_2'\in fp_1$ so that $d(p_1',p_2')\leqslant H(fp_o,fp_1)+q$. Suppose that $p_2'\in K$. Then

$p_2' \in K \cap f(K) \subseteq TK$ which implies that there exists $p_2 \in K$ such that $Tp_2 = p_2'$. If $p_2' \notin K$ then there exists $q \in \partial K$ so that:

$$d(Sp_1, q) + d(q, p_2') = d(Sp_1, p_2').$$

Since $q \in \partial K \subseteq TK$ there exists $p_2 \in K$ such that $q = Tp_2$ and so:

$$d(Sp_1, Tp_2) + d(Tp_2, p_2') = d(Sp_1, p_2').$$

Let $p_3' \in fp_2$ be such that:

$$d(p_2', p_3') \leqslant H(fp_1, fp_2) + q^2.$$

It is easy to see that in this way we obtain two sequences $\{p_n\}$, $n \in N$ and $\{p_n'\}$, $n \in N$ such that

1. For every $n \in N$: $p_n' \in fp_{n-1}$
2. For every $n \in N$: $p_{2n}' \in K \Rightarrow p_{2n}' = Tp_{2n}$
   $p_{2n}' \notin K \Rightarrow Tp_{2n} \in \partial K$ and

(1) $$d(Sp_{2n-1}, Tp_{2n}) + d(Tp_{2n}, p_{2n}') = d(Sp_{2n-1}, p_{2n}').$$

3. For every $n \in N$: $p_{2n+1}' \in K \Rightarrow p_{2n+1}' = Sp_{2n+1}$
   $p_{2n+1}' \notin K \Rightarrow Sp_{2n+1} \in \partial K$ and:

(2) $$d(Tp_{2n}, Sp_{2n+1}) + d(Sp_{2n+1}, p_{2n+1}') = d(Tp_{2n}, p_{2n+1}').$$

4. For every $n \in N$:

$$d(p_n', p_{n+1}') \leqslant H(fp_{n-1}, fp_n) + q^n.$$

Let P_o, P_1, Q_o and Q_1 be defined by:

$$P_o = \{p_{2n},\ n \in N \text{ and } p_{2n}' = Tp_{2n}\}$$

$$P_1 = \{p_{2n},\ n \in N \text{ and } p_{2n}' \neq Tp_{2n}\}$$

$$Q_o = \{p_{2n+1},\ n \in N \text{ and } p_{2n+1}' = Sp_{2n+1}\}$$

$$Q_1 = \{p_{2n+1},\ n \in N \text{ and } p_{2n+1}' \neq Sp_{2n+1}\}.$$

First we shall prove that:

$$(p_{2n}, p_{2n+1}) \notin P_1 \times Q_1 \text{ and } (p_{2n-1}, p_{2n}) \notin Q_1 \times P_1.$$

If $p_{2n} \in P_1$ then $p_{2n}' \neq Tp_{2n}$ and in this case we have that $Tp_{2n} \in \partial K$ which implies that $p_{2n+1}' \in fp_{2n} \subseteq K$. Hence $p_{2n+1}' = Sp_{2n+1}$ which means that $p_{2n+1} \in Q_o$. We can prove similarly that $(p_{2n-1}, p_{2n}) \notin Q_1 \times P_1$.

Let us prove that for every $n \in N$:

$$(3)\qquad d(Tp_{2n},Sp_{2n+1}) \leq \begin{cases} qd(Sp_{2n-1},Tp_{2n}) + q^{2n} \\ \quad\text{or} \\ qd(Sp_{2n-1},Tp_{2n-2}) + q^{2n-1} + q^{2n} \end{cases}$$

and

$$(4)\qquad d(Sp_{2n-1},Tp_{2n}) \leq \begin{cases} qd(Sp_{2n-1},Tp_{2n-2}) + q^{2n-1} \\ \quad\text{or} \\ qd(Sp_{2n-3},Tp_{2n-2}) + q^{2n-2} + q^{2n-1} . \end{cases}$$

If $(p_{2n},p_{2n+1}) \in P_o \times Q_o$ then:

$$d(Tp_{2n},Sp_{2n+1}) = d(p'_{2n},p'_{2n+1}) \leq H(fp_{2n-1},fp_{2n}) + q^{2n} \leq$$
$$\leq qd(Sp_{2n-1},Tp_{2n}) + q^{2n} .$$

Suppose that $(p_{2n},p_{2n+1}) \in P_o \times Q_1$. Then from (2) we obtain that:

$$d(Tp_{2n},Sp_{2n+1}) \leq d(Tp_{2n},p'_{2n+1}) = d(p'_{2n},p'_{2n+1}) \leq$$
$$\leq qd(Sp_{2n-1},Tp_{2n}) + q^{2n} .$$

From the relation $(p_{2n},p_{2n+1}) \in P_1 \times Q_o$ we have that:

$$d(Tp_{2n},Sp_{2n+1}) \leq d(Tp_{2n},p'_{2n}) + d(p'_{2n},p'_{2n+1}) \leq$$
$$\leq d(Tp_{2n},p'_{2n}) + qd(Sp_{2n-1},Tp_{2n}) + q^{2n} \leq$$
$$\leq d(Tp_{2n},p'_{2n}) + d(Sp_{2n-1},Tp_{2n}) + q^{2n} .$$

Using (1) we obtain that:

$$d(Tp_{2n},Sp_{2n+1}) \leq d(Sp_{2n-1},p'_{2n}) + q^{2n} .$$

Since $p_{2n} \in P_1$ and $(p_{2n-1},p_{2n}) \notin Q_1 \times P_1$ we obtain that $p_{2n-1} \in Q_o$ which means that $p'_{2n-1} = Sp_{2n-1}$ and so:

$$d(Tp_{2n},Sp_{2n+1}) \leq d(p'_{2n-1},p'_{2n}) + q^{2n} \leq$$
$$\leq qd(Sp_{2n-1},Tp_{2n-2}) + q^{2n-1} + q^{2n} .$$

If $(p_{2n-1},p_{2n}) \in Q_o \times P_o$ then:

$$d(Sp_{2n-1},Tp_{2n}) = d(p'_{2n-1},p'_{2n}) \leq qd(Sp_{2n-1},Tp_{2n-2}) + q^{2n-1} .$$

Suppose that $(p_{2n-1},p_{2n}) \in Q_o \times P_1$. Then from (1) we have that:

$$d(Sp_{2n-1},Tp_{2n}) \leq d(Sp_{2n-1},p'_{2n}) = d(p'_{2n-1},p'_{2n}) \leq$$
$$\leq qd(Sp_{2n-1},Tp_{2n-2}) + q^{2n-1} .$$

If $(p_{2n-1},p_{2n}) \in Q_1 \times P_o$ then:

$$d(Sp_{2n-1},Tp_{2n}) \leqslant d(Sp_{2n-1},p'_{2n-1}) + d(p'_{2n-1},Tp_{2n}) =$$
$$= d(Sp_{2n-1},p'_{2n-1}) + d(p'_{2n-1},p'_{2n}) \leqslant$$
$$\leqslant d(Sp_{2n-1},p'_{2n-1}) + qd(Sp_{2n-1},Tp_{2n-2}) + q^{2n-1} \leqslant$$
$$\leqslant d(Sp_{2n-1},p'_{2n-1}) + d(Sp_{2n-1},Tp_{2n-2}) + q^{2n-1}.$$

Since $p_{2n-1} \in Q_1$ from (2) we obtain that:

$$d(Sp_{2n-1},Tp_{2n}) \leqslant d(Tp_{2n-2},p'_{2n-1}) + q^{2n-1}.$$

Further from $p_{2n-1} \in Q_1$ it follows that $p_{2n-2} \in P_o$ which means that $Tp_{2n-2} = p'_{2n-2}$.

Hence:

$$d(Sp_{2n-1},Tp_{2n}) \leqslant d(p'_{2n-2},p'_{2n-1}) + q^{2n-1} \leqslant$$
$$\leqslant qd(Sp_{2n-3},Tp_{2n-2}) + q^{2n-2} + q^{2n-1}.$$

Using the above inequalities we conclude that (3) and (4) hold. Inequalities (3) and (4) can be written in the form:

$$z_{2n} = Tp_{2n}, z_{2n+1} = Sp_{2n+1} \quad (n \in N)$$

$$d(z_n,z_{n+1}) \leqslant \begin{cases} qd(z_n,z_{n-1}) + q^n \\ \quad\text{or} \\ qd(z_{n-2},z_{n-1}) + q^n + q^{n-1}, \end{cases}$$

which is inequality (*) from [1].

Then:
$$d(z_n,z_{n+1}) \leqslant q^{\frac{n}{2}}(\delta + n),$$

where
$$\delta = q^{-\frac{1}{2}} \max\{d(z_o,z_1),d(z_1,z_2)\}.$$

Hence the sequence $\{z_n\}_{n \in N}$ is a Cauchy sequence and let

$$z = \lim_{n\to\infty} z_n = \lim_{n\to\infty} Tp_{2n} = \lim_{n\to\infty} Sp_{2n+1}.$$

There exists at least one subsequence $\{p_{2n_k}\}_{k \in N}$ or $\{p_{2n_k+1}\}_{k \in N}$ which is contained in P_o or Q_o respectively since

$$(p_{2n},p_{2n+1}) \notin P_1 \times Q_1 \quad \text{and} \quad (p_{2n-1},p_{2n}) \notin Q_1 \times P_1.$$

Suppose that there exists $\{p_{2n_k}\}_{k \in N}$ such that $p_{2n_k} \in P_o$ for every $k \in N$. Then

$$p'_{2n_k} = Tp_{2n_k} \in fp_{2n_k-1} \quad (k \in N).$$

Let us prove that $Sz \in fz$. Using the weak commutativity of the pair (f,S) and the relations:

$$Tp_{2n_k} \in fp_{2n_k-1} \cap K \quad \text{and} \quad Sp_{2n_k-1} \in K, \quad k \in N$$

we obtain that:

$$d(STp_{2n_k}, fSp_{2n_k-1}) \leq d(fp_{2n_k-1}, Sp_{2n_k-1}) \leq$$

$$\leq d(Tp_{2n_k}, Sp_{2n_k-1}).$$

Since $\lim\limits_{k\to\infty} d(Tp_{2n_k}, Sp_{2n_k-1}) = 0$, we obtain that

$$\lim_{k\to\infty} d(STp_{2n_k}, fSp_{2n_k-1}) = 0.$$

From the inequality:

$$d(STp_{2n_k}, fz) \leq d(STp_{2n_k}, fSp_{2n_k-1}) + H(fSp_{2n_k-1}, fz),$$

since f is H-continuous, we obtain that:

$$\lim_{k\to\infty} d(STp_{2n_k}, fz) = 0.$$

Hence from the inequality:

$$d(Sz, fz) \leq d(Sz, STp_{2n_k}) + d(STp_{2n_k}, fz)$$

using the continuity of the mapping S we obtain that $d(Sz,fz) = 0$ and so $Sz \in fz$, which implies that $\{Tz, Sz\} \cap fz \neq \emptyset$.

Suppose now that $S,T: M \to M$ and that *(i)* and *(ii)* hold. From 4. we obtain that for every $k \in N$:

$$d(p'_{2n_k+1}, p'_{2n_k}) \leq H(fp_{2n_k}, fp_{2n_k-1}) + q^{2n_k} \leq$$

$$\leq qd(Tp_{2n_k}, Sp_{2n_k-1}) + q^{2n_k}$$

which implies that:

$$d(p'_{2n_k+1}, Tp_{2n_k}) \leq qd(Tp_{2n_k}, Sp_{2n_k-1}) + q^{2n_k}.$$

Since $\lim\limits_{k\to\infty} d(Tp_{2n_k}, Sp_{2n_k-1}) = 0$, we obtain that:

$$\lim_{k\to\infty} p'_{2n_k+1} = z$$

and so :

(5) $$Tz = T(\lim_{k\to\infty} p'_{2n_k+1}) = \lim_{k\to\infty} Tp'_{2n_k+1}.$$

Using the implication *(i)* for $x = p_{2n_k}$ and $y = p'_{2n_k+1}$ we conclude that:

$$d(Tp'_{2n_k+1}, fTp_{2n_k}) \leq d(fp_{2n_k}, Tp_{2n_k}) \leq d(p'_{2n_k+1}, Tp_{2n_k})$$

and since $\lim\limits_{k\to\infty} d(p'_{2n_k+1}, Tp_{2n_k}) = d(z,z) = 0$, we have that

$$\lim_{k\to\infty} d(Tp'_{2n_k+1}, fTp_{2n_k}) = 0.$$

Further:

$$d(Tp'_{2n_k+1},fz) \leq d(Tp'_{2n_k+1},fTp_{2n_k}) + H(fTp_{2n_k},fz)$$

and since $\lim_{k\to\infty} H(fTp_{2n_k},fz) = 0$, it follows that:

(6) $$\lim_{k\to\infty} d(Tp'_{2n_k+1},fz) = 0 .$$

Using (5) and (6) we obtain that $d(Tz,fz) = 0$ since:

$$d(Tz,fz) \leq d(Tz,Tp'_{2n_k+1}) + d(Tp'_{2n_k+1},fz) .$$

From $d(Tz,fz) = 0$ we conclude that $Tz \in fz$.

COROLLARY (THEOREM A) [1] *Let* (M,d) *be a complete convex metric space,* K *a nonempty closed subset of* M, $f:K\to CB(M)$ *so that for every* $x \in \partial K$, $fx \in K$ *and:*

$$H(fx,fy) \leq qd(x,y), \text{ for every } x,y \in K$$

where $q \in (0,1)$. *Then there exists* $z \in K$ *such that* $z \in fz$.

PROOF: It is obvious that for $S = T = Id|K$ all the conditions of Theorem 1 are satisfied.

3. A COMMON FIXED POINT THEOREM IN METRIC SPACES WITH A CONVEX STRUCTURE

In this section we shall need the following definition which is given in [5].

DEFINITION 1. *Let* (X,d) *be a metric space,* $A,S:X\to 2^X$ *and* $K\subseteq X$. *The mapping* A *is said to be* (α,S)*-densifying on the set* K *if and only if for every* $M\subseteq K$ *such that* $S(M),A(M) \in B(X)$ *the following implication holds:*

$$\alpha(S(M)) \leq \alpha(A(M)) \implies \overline{M} \text{ is compact.}$$

REMARK In [4] the definition of the Palais-Smale condition for the singlevalued mappings is given. We shall recall this definition. Let X and Γ be metric spaces and $p,q:\Gamma\to X$. The pair (p,q) satisfies the Palais-Smale condition if for every sequence $\{y_n\}_{n\in N}$ from Γ the relation

$$\lim_{n\to\infty} d_X(p(y_n),q(y_n)) = 0$$

implies that there exists a convergent subsequence $\{y_{n_k}\}_{k\in N}$ of the sequence $\{y_n\}_{n\in N}$.

From this definition it is obvious that if in Definition 1 the mappings S and A are singlevalued and $A(K)$ is bounded, the following implication holds:
A is (α,S) densifying on $K \Rightarrow$ the pair (A,S) satisfies the Palais-Smale condition on K (i.e. the pair $(A|_K,S|_K)$ satisfies the Palais-Smale condition, where $A|_K$ and $S|_K$ are the restrictions of A and S on K respectively).

Indeed, if A is (α,S) densifying on K and $A(K)$ is bounded then from:

$$\lim_{n\to\infty} d_X(S(y_n),A(y_n)) = 0$$

it follows that $\alpha(SL)=\alpha(AL)$ (this can be proved easily), where $L=\{y_n,\ n\in N\}$, which implies that $\overline{L}$ is compact. Hence the pair $(A|_K,S|_K)$ satisfies the Palais-Smale condition.

Let us remark that in [4] the notion of a condensing pair (p,q) of singlevalued mappings is given, which is similar to Definition 1 if the mappings A and S are singlevalued. If $q:\Gamma\to 2^X$ and $p:\Gamma\to X$, we can introduce the Palais-Smale condition in the following way.

DEFINITION 2. *The pair (p,q) satisfies the Palais-Smale condition if for every sequence $\{y_n\}_{n\in N}$ from Γ, the relation:*

$$\lim_{n\to\infty} d_X(p(y_n),v_n) = 0\,,\ \textit{for some}\ \{v_n\}_{n\in N}$$

such that $v_n\in q(y_n)$ $(n\in N)$, implies the existence of a convergent subsequence $\{y_{n_k}\}_{k\in N}$ of the sequence $\{y_n\}_{n\in N}$.

The set of coincidence points for the pair (p,q) is defined by:

$$K(p,q) = \{y,\ y\in\Gamma\ \text{and}\ p(y)\in q(y)\}.$$

and for $\varepsilon>0$ we shall define an ε-coincidence point for the pair (p,q) in the following way.

DEFINITION 3. *A point $y\in\Gamma$ is an ε-coincidence point for the pair (p,q) if there exists an element $v\in q(y)$ such that* $d_X(p(y),v)<\varepsilon$.

Similarly as in [4] (Proposition 2.3) we can prove the following proposition.

PROPOSITION: *Let (Γ,d_Γ) and (X,d_X) be metric spaces, $q:\Gamma\to 2^X$ a closed mapping and $p:\Gamma\to X$ a continuous map-*

ping. Let for every $\varepsilon > 0$ the pair (p,q) have an ε-coincidence point and satisfy the Palais-Smale condition. Then

$$K(p,q) \neq \emptyset .$$

PROOF: Let $\varepsilon_n = \frac{1}{n}$ $(n \in N)$ and y_n be an ε_n-coincidence point for the pair (p,q). Let $v_n \in q(y_n)$, $(n \in N)$ so that $d_X(p(y_n), v_n) < \frac{1}{n}$. Then

$$\lim_{n\to\infty} d_X(p(y_n), v_n) = 0$$

and since the pair (p,q) satisfies the Palais-Smale condition, there exists a convergent subsequence $\{y_{n_k}\}_{k \in N}$ of the sequence $\{y_n\}_{n \in N}$. Let $\lim_{k\to\infty} y_{n_k} = y$. Since $\lim_{k\to\infty} d_X(p(y_{n_k}), v_{n_k}) = 0$, it follows that

$$\lim_{k\to\infty} v_{n_k} = \lim_{k\to\infty} p(y_{n_k}) = p(y) .$$

From $v_{n_k} \in q(y_{n_k})$, $(k \in N)$, using the closedness of the mapping q, we obtain that $p(y) \in q(y)$, which means that $y \in K(p,q)$.

Similarly as in [5] we shall prove a common fixed point theorem if:

$$H(fx,fy) \leq d(Sx,Ty) \text{, for every } x,y \in K,$$

where K is (W,x_o)-star convex $(x_o \in K$ and $W(K,x_o,(0,1)) \subset K)$

THEOREM 2. *Let (M,d) be a complete metric space with a convex structure W with property II, K a nonempty closed (W,x_o) - star convex subset of M, $S,T : M \to M$ continuous (W,x_o) convex mappings, $f : K \to k(M)$ (the family of all nonempty compact subsets of M) so that the following conditions are satisfied:*

1. fK *is bounded,* $K \subseteq SK \cap TK$ *and:*
$$Tx \in \partial K \Rightarrow fx \in K \text{ ; } Sx \in \partial K \Rightarrow fx \in K .$$
2. *For every* $x,y \in K$:
$$H(fx,fy) \leq d(Sx,Ty) .$$
3. *The mapping* f *is* H*-continuous and:*
$$Sx \in K \Rightarrow fSx = Sfx \text{ ; } Tx \in K \Rightarrow Tfx = fTx .$$
4. *The mapping* f *is* (α,S) *or* (α,T) *densifying.*

Then there exists $z \in K$ *so that* $Tz \in fz$ *and* $Sz \in fz$.

PROOF: As in [5] let $\{k_n\}_{n \in N}$ be such a sequence from $(0,1)$ that $\lim_{n\to\infty} k_n = 1$ and $f_n : K \to k(M)$ be defined by:

$f_n x = W(fx, x_o, k_n)$, for every $x \in K$ and every $n \in N$. We shall prove that for every $n \in N$ there exists $x_n \in K$ so that $Tx_n \in f_n x_n$ and $Sx_n \in f_n x_n$. It remains to be proved that T, S and f_n satisfy all the conditions of Theorem 1. Since the set K is (W, x_o)-star convex from the condition 1. of this theorem we obtain that

$$Tx \in \partial K \Rightarrow f_n x \in K; \quad Sx \in \partial K \Rightarrow f_n x \in K.$$

Further $f_n Sx$, if $Sx \in K$, is equal to the set $Sf_n x$ since:

$$f_n Sx = \bigcup_{z \in fSx} W(z, x_o, k_n) = \bigcup_{z \in Sfx} W(z, x_o, k_n) =$$

$$= \{W(Sy, x_o, k_n), y \in fx\} = \{SW(y, x_o, k_n), y \in fx\} = Sf_n x, \text{ for every } n \in N.$$

Similarly from $Tx \in K$ we obtain that $f_n Tx = Tf_n x$, for every $n \in N$. It is easy to see that from $f_n Sz = Sf_n z$ we obtain the inequality:

$$d(Sy, f_n Sz) \leq d(f_n z, Sz), \text{ for } y \in f_n z$$

since

$$d(Sy, f_n Sz) = \inf_{z \in f_n Sz} d(Sy, z) = \inf_{z \in Sf_n z} d(Sy, z) <$$

$$\leq d(Sy, Sy) = 0 \leq d(f_n z, Sz).$$

Since the convex structure W satisfies condition II it follows that:

$$H(f_n x, f_n y) = H(W(fx, x_o, k_n), W(fy, x_o, k_n)) \leq$$

$$\leq k_n H(fx, fy) \leq k_n \, d(Sx, Ty)$$

for every $x, y \in K$, and hence the mapping f_n is H-continuous. Further, from the compactness of fx, for every $x \in K$ and the continuity of W in respect to the first variable it follows that $f_n x$ is compact for every $x \in K$. From $K \subseteq SK \cap TK$ we obtain that $f_n K \cap K \subseteq SK \cap TK$. Hence, all the conditions of Theorem 1. are satisfied and for every $n \in N$ there exists $x_n \in K$ so that $Tx_n \in f_n x_n$ and $Sx_n \in f_n x_n$. The rest of the proof is as in the proof of Theorem 2. from [5]. For the completeness we shall give the rest of the proof. Since $Tx_n \in f_n x_n$ and $Sx_n \in f_n x_n$ $(n \in N)$ there exist $u_m \in fx_m$ and $v_m \in fx_m$ $(m \in N)$ so that $Sx_m = W(u_m, x_o, k_m)$ and $Tx_m = W(v_m, x_o, k_m)$. Then it is easy to prove that:

(7) $$\lim_{m \to \infty} d(Sx_m, u_m) = \lim_{m \to \infty} d(Tx_m, v_m) = 0.$$

Hence, for every $\varepsilon > 0$ there exists an ε-coincidence point for (S, f) and (T, f). Since the mapping f is (α, S) or (α, T) densifying on the set K and the set $f(K)$ is bounded, we conclude that the pair (S,f) or the pair (T,f) satisfies the Palais--Smale condition on the set K. Further, the mapping f is H-continuous and fx is compact for every $x \in K$ and so the mapping f is closed. Using the Proposition we conclude that $K(S,f) \neq \emptyset$ or $K(T,f) \neq \emptyset$. From the Proposition it follows that $y \in K(S,f)$ (if, for example $K(S,f) \neq \emptyset$) is of the form $y = \lim_{k\to\infty} x_{n_k}$, for some subsequence $\{x_{n_k}\}_{k\in N}$ of the sequence $\{x_n\}_{n\in N}$. From (7) we have that $\lim_{k\to\infty} v_{n_k} = Ty \in fy$ and so

$$y \in K(T,f) \cap K(S,f) .$$

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REZIME

TAČKE KOINCIDENCIJE VIŠEZNAČNIH PRESLIKAVANJA U KONVEKSNIM METRIČKIM PROSTORIMA

U ovom je radu dokazano uopštenje dobro poznate teoreme o nepokretnoj tački Assada i Kirka za višeznačna preslikavanja u konveksnim metričkim prostorima. Višeznačna verzija uslova Palais-Smalea je uvedena i primenjena u dokazu Teoreme 2, koja sadrži jedan rezultat o postojanju tačke koincidencije za višeznačna preslikavanja u metričkim prostorima sa konveksnom strukturom.

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