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ON INJECTIVE MODULES AND GENERALIZATIONS

Roger Yue Chi Ming

Université Paris VII, U.E.R. de Mathématique et Informatique, 2, Place Jussieu 75251 Paris Cedex 05, France

(Dedicated to Professor Yuzo Utumi on his sixtieth birthday)

ABSTRACT

In this paper two generalizations of injectivity are introduced and used to characterize some well-known classes of rings.

INTRODUCTION

Throughout, A denotes an associative ring with identity and A-modules are left unital, unless otherwise stated. J, Z stand respectively, for the Jacobson radical and left singular ideal of A. Two generalizations of injectivity, called CY and KY-injectivity, are introduced to study von Neumann regular and Noetherian rings. Conditions are given for two modules to have isomorphic injective hulls. This note contains the following results: (1) If M is a CY-injective module, then any cyclic submodule has an injective hull in M; (2) If A has a classical left quotient ring Q such that every

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divisible torsionfree A-module is CY-injective, then Q is semi-simple Artinian; (3) The following conditions are equivalent for a left non-singular ring A: (a) A is left Noetherian; (b) every CY-injective A-module is KY-injective; (c) every CY-injective A-module is injective; (4) For any left KY-injective ring A, A/J is von Neumann regular and J = Z; (5) If A is semi-primary, M, N A-modules such that either $r_M(Z)$ is isomorphic to $r_N(Z)$ or $r_M(J)$ is isomorphic to $r_N(J)$ (as left A-modules), then M and N have isomorphic injective hulls. It is also shown that in certain situations, proper direct summands of semi-prime right KY-injective rings possess non-trivial central idepotents.

For any A-module M, Z(M) = {z ∈ M/Lz = 0 for some essential left ideal L of A} is the left singular submodule of M and M is called singular (resp. non-singular) if Z(M) = = M (resp. Z(M) = 0). Thus A is left non-singular iff Z = 0. An A-module M is called divisible if M = cM for each non-zero-divisor c of A. M is called torsionfree if cy ≠ 0 for every non-zero-divisor c of A and non-zero element y of M. Recall that A-module M is p-injective if, for any principal left ideal P of A, every left A-homomorphism of P into M extends to one of A into M. Then A is von Neumann regular iff every left (right) A-module is p-injective. Note that p-injective modules need not be flat and the converse is not true either. However, if I is a p-injective left ideal of A, then A/I is a flat A-module. If M is a maximal left ideal which is a two-sided ideal of A, then ${}_A A/M$ is flat iff A/M_A is injective iff A/M_A is p-injective. As usual, A is called a left V-ring if every simple left A-module is injective. Injective modules have been extensively studied by many authors since several years (cf. for example, [2], [3]). We now introduce the following two generalizations of injectivity, the first one being motivated by p-injectivity.

Definitions (1) *An A-module* M *is called* CY-*injective if, for any A-module* Y, *any cyclic submodule* C *of* Y, *every left* A*-homomorphism of* C *into* M *extends to one of* Y *into* M;

(2) *An* A*-module* Y *is called* KY*-injective if, for any complement submodule* K *of* Y*, any left* A*-monomorphism* $g : K \to Y$ *and left* A*-homomorphism* $F : K \to Y$*, there exists an endomorphism* h *of* Y *such that* $hg = f$.

It is easily seen that any direct summand of a KY-injective A-module is KY-injective.

CY-injectivity and KY-injectivity are distinct effective generalizations of injectivity. Recall that an A-module M is continuous if every submodule isomorphic to a complement submodule of M is a direct summand of M (cf. [6]). Continuous modules generalize quasiinjective modules. Since a continuous A-module is KY-injective, it follows that KY-injectivity does not imply CY-injectivity (cf. [1] and Proposition 8 below). The converse is not true either (cf. Theorem 5).

We start with various properties of CY-injective modules. Obviously, CY-injectivity implies p-injectivity but the converse is not true, as shown by our first proposition.

Proposition 1. *Let* M *be a* CY*-injective* A*-module. Then any cyclic submodule has an injective hull in* M*. In particular, every cyclic* CY*-injective* A*-module is injective.*

Proof. Let C be a cyclic submodule of M, E an injective hull of C. If g, j are the inclusion maps of C into M and C into E respectively, there exists a left A-homomorphism $h : E \to M$ such that $hj = g$. For any $d \in \ker h \cap C$, $d = g(d) = hj(d) = h(d) = 0$ and since C is an essential left submodule of E, then $\ker h = 0$ which implies that h is a monomorphism, whence $h(E)$ ($\approx E$) is an injective A-module contained in M. This shows that C has an injective hull contained in M (because $C \subseteq h(E)$). In case M is cyclic, then it is obvious that M is injective.

Corollary 1.1. (a) A *is a left* V-*ring iff every simple* A-*module is* CY-*injective;*

(b) A *is left self-injective regular iff every finitely generated left ideal of* A *is a* CY-*injective* A-*module.*

Recall that a ring Q is a classical left quotient ring of A if

(i) $A \subseteq Q$;

(ii) every non-zero-divisor of A is invertible in Q;

(iii) every element of Q is of the form $q = b^{-1}a$, $a, b \in A$, b being a non-zero-divisor.

As usual, a left (right) ideal of A is called reduced if it contains no non-zero nilpotent element.

Remark 1. (a) Let A have a classical left quotient ring Q. Then Q is injective iff Q is CY-injective. Consequently, if A is left non-singular with ${}_AQ$ CY-injective, then Q is left self-injective regular and is the maximal left quotient ring of A;

(b) If A is left Noetherian such that each prime factor ring contains a non-zero CY-injective left ideal, then A is left Artinian;

(c) A is a division ring iff A is a prime ring containing a non-zero reduced CY-injective left ideal (cf. [8, Proposition 6]).

Proposition 2. *A direct sum of* A-*modules is* CY-*injective if and only if each direct summand is* CY-*injective.*

Proof. Given $M = \bigoplus_{i \in I} M_i$, where each $M_i (i \in I)$ is a CY-injective A-module, we prove that M is Cy-injective. Let N be an A-module, $c \in N$, $f : Ac \to M$ a left A-homomorphism, $y = f(c) = y_{i1} + y_{i2} + \ldots + y_{ip}$, $y_{ij} \in M_{ij}$, $1 \leq j \leq r$. If $p_j : M \to M_{ij}$ is the natural projection for each j, $1 \leq j \leq r$, then $p_j \ f : Ac \to M_{ij}$ and since M_{ij} is CY-injective, there exists a left A-homomorphism $h_j : N \to M_{ij}$ which extends $p_j f$. Define $h : N \to M$ by $h(u) = h_1(u) + \ldots + h_r(u)$ for all $u \in N$. Then $h(c) = p_1 f(c) + \ldots + p_r f(c) = y_{i1} + \ldots + y_{ir} = y$ which shows that h extends f to N. This proves that M is CY-injective. Conversely, using the natural injection and projection, it is easily seen that a direct summand of a CY-injective A-module is CY-injective.

It is well-known that A is left Noetherian iff any direct sum of injective A-modules is injective.

Corollary 2.1. *If every CY-injective* A*-module is injective, then* A *is left Noetherian. Consequently,* A *is a principal left ideal ring iff every finitely generated left ideal of* A *is principal and every CY-injective* A*-module is injective.*

Corollary 2.2. *The following conditions are equivalent:* (a) A *is a left Noetherian left* V*-ring whose quasi-injective and CY-injective modules are injective;*

(b) *An* A*-module is quasi-injective iff it is CY-injective. (cf.* [2, *Proposition 20.4B*]*).*

Remark 2. If A has non-zero left socle S, than S is a CY-injective A-module iff every minimal left ideal is injective. Therefore, A is simple Artinian iff A is prime with a non-zero socle which is a left and right CY-injective A-module.

Remark 3. If A is von Neumann regular, then every cyclic submodule of a projective CY-injective A-module is injective.

Proposition 3. *Let* A *have a classical left quotient ring* Q*. If every divisible torsionfree* A*-module is CY-injective, then* Q *is semi-simple Artinian.*

Proof. Let $C = Qy$ be a cyclic Q-module. It is sufficient to prove that C is a direct summand of every Q-module M containing it. Then every cyclic Q-module will be injective and the proposition will follow from [4, Theorem 3.2]. Since C is a torsionfree divisible A-module and ${}_AAy$ is essential in ${}_AC$, then C is injective (cf. the proof of Proposition 1). Therefore $M = C \oplus P$ and since M is divisible, then so is P. For any $y \in P$, any $q \in Q$, $q = b^{-1}d$, $b, d \in A$, if $u = qy = b^{-1}w$, where $w = dy$, since $P = bP$, then $w = bv$ for some $v \in P$ and hence $bu = w = bv$ which implies $u = v \in P \subseteq M$, showing that P is a left Q-module. Thus ${}_QM = {}_QC \oplus {}_QP$ which proves that C is an injective left Q-module.

Corollary 3.1. *If A has a von Neumann regular clasical left quotient ring Q and every p-injective torsionfree A-module is CY-injective, then Q is semi-simple Artinian.*

As usual, A is called left duo if every left ideal is a two-sided ideal.

Corollary 3.2. *A left duo ring whose divisible torsionfree left modules are CY-injective possesses a classical left quotient ring which is a finite direct sum of division rings.*

Proposition 4. *Let A be left non-singular such that every direct sum of the injective hulls of cyclic singular A-modules is injective. Then the singular submodule of any CY-injective A-module is injective.*

Proof. Let M be a CY-injective A-module with $Z(M) \neq 0$. If $0 \neq z \in Z(M)$, then Az has an injective hull U contained in M by Proposition 1 and since A is left non-singular, we know that U must be contained in Z(M). Let E denote the set of the injective hulls of all cyclic singular A-modules contained in M. Then the set F of all independent families $\{N_j\}$ of elements of E is an inductive set and by Zorn's Lemma, F has a maximal member $\{N_i\}_{i\in I_o}$. Now $K = \bigoplus_{i\in I} N_i \subseteq Z(M)$ and $_AK$ is injective by hypothesis, which implies $Z(M) = K \oplus V$. If $0 \neq w \in V$, then Aw has an injective hull W contained in Z(M). Then $W \cap K = 0$ yields a member of F which strictly contains $\{N_i\}_{i\in I_o}$, contradicting its maximality in F. Thus $V = 0$ and $Z(M) = K$ is injective.

Applying [7, Proposition 4], we get

Corollary 4.1. *If* A *is a left Noetherian ring whose divisible singular modules are* CY*-injective, then* A *is left hereditary.*

The next result connects CY-injectivity with KY-injectivity.

Theorem 5. *The following conditions are equivalent for a left non-singular ring* A*:*

(1) A *is left Noetherian;*

(2) A *is of left finite Goldie dimension and every direct sum of the injective hulls of cyclic singular* A*-modules is injective;*

(3) *Every* CY*-injective* A*-module is* KY*-injective;*

(4) *Every* CY*-injective* A*-module is injective.*

Proof. Obviously, (1) implies (2).

Assume (2). Let M be a CY-injective A-module. By Proposition 4, $M = Z(M) \oplus Q$, where $_AZ(M)$ is injective and $_AQ$ is non-singular. Since the injective hull of a non-singular A-module is non-singular, by [5, Theorem 2.5], any direct sum of the injective hulls of cyclic non-singular A-modules is injective and non-singular. As in the proof of Proposition 4, it can be shown that $_AQ$ (which is also CY-injective) is a direct sum of injective hulls of cyclic non-singular left submodules, whence $_AQ$ is injective and (2) implies (3).

Assume (3). If $_AM$ is CY-injective, E the injective hull of $_AM$, set $S = {_AM} \oplus {_AE}$. Then $_AS$ is CY-injective (Proposition 2) and is therefore KY-injective by hypothesis. If $i : M \to S$, $p : S \to M$ are the natural injection and projection (pi is therefore the identity map on M), $j : M \to E$, $u : E \to S$ the inclusion maps, then there exists a map $h : S \to S$ such that $huj = i$. With $phu = q$, we have a left A-homomorphism $q : E \to M$ such that $qj = phuj = pi$, the identity map on M, which proves that $_AM$ is a direct summand of $_AE$. Thus M = E and (3) implies (4).

(4) implies (1) by Corollary 2.1.

Using [2, Theorem 24.20], the next result may be similarly proved.

Proposition 6. *The following conditions are equivalent:*

(1) A *is quasi-Frobeniusean;*

(2) *The direct sum of an injective and a projective A-modules is* KY*-injective.*

Looking at Theorem 5, we may ask the following: when are KY-injective A-modules injective? The next KY-injective analogue of [2, Proposition 20.4B] holds.

Remark 4. The following conditions are equivalent: (1) Every KY-injective A-module is injective; (2) The direct sum of any two KY-injective A-modules is KY-injective.

Remark 5. (1) If A is of left finite Goldie dimension such that every divisible singular A-module is injective, then A is left hereditary, left Noetherian; (2) The following are equivalent: (a) An A-module is CY-injective iff it is KY-injective; (b) All CY-injective and KY-injective A-modules are injective; (3) If the sum of any two KY-injective A-modules is KY-injective then A is a left hereditary, left Noetherian, left V-ring.

Note that if $_AM$ is KY-injective and N is a left submodule of M, then $_AN$ is a direct summand of $_AM$ iff N is a complement left submodule of M which is KY-injective.

A is called left KY-injective if ${}_AA$ is KY-injective. An element c of A is called left regular if l(c) = 0. Then c is a non-zero-divisor iff it is left and right regular. The next result extends [6, Lemma 4.1].

Proposition 7. *Let* A *be a left* KY*-injective ring. Then* (1) *Every left regular element is right invertible in* A*; consequently, every left (right)* A*-module is divisible;* (2) Z = J *and* A/Z *is von Neumann regular.*

Proof. (1) Let c ∈ A such that l(c) = 0. Since the map g : A → A given by g(a) = ac (a ∈ A) is a monomorphism, then with i : A → A the identity map, there exists a left A-homomorphism h : A → A such that hg = i, which yields 1 = hg(1) = h(c) = ch(1), showing that any left regular element is right invertible. It follows that every non-zero-divisor is invertible and every left (right) A-module is divisible.

(2) For any z ∈ Z, a ∈ A, since l(za) ∩ l(1-za) = = 0, then l(1-za) = 0 which implies 1 - za right invertible in A by (1), showing that z ∈ J. In order to have Z = J, since (J + Z)/Z is contained in the Jacobson radical of A/Z, it is sufficient to prove that A/Z is von Neumann regular. Let $0 \neq \bar{b} \in A/Z$, $\bar{b} = b + Z$, b ∈ A, b ∉ Z. There exists a non-zero complement left ideal C of A such that E = C ⊕ l(b) is an essential left ideal. For any 0 ≠ c ∈ C, cb ≠ 0 and since the map g : C → A given by g(c) = cb(c ∈ C) is a monomorphism, if i : C → A is the inclusion map, then there exists a left A-homomorphism h : A → A such that hg = i. For every c ∈ C, c = hg(c) = h(cb) = cbd, where d = h(1). Therefore C ⊆ l(b-bdb) which yields E ⊆ l(b-bdb), whence $\bar{b} = \bar{b}\bar{d}\bar{b}$ in A/Z, proving that A/Z is von Neumann regular.

Corollary 7.1. A *is left continuous regular iff* A *is a left non-singular left* KY*-injective ring whose complement left ideals are finitely generated.*

Corollary 7.2. *A left Noetherian left KY-injective ring is left Artinian.*

Question: When is a left Noetherian left KY-injective ring quasi-Frobeniusean?

T is called a strongly regular ideal of A if T is a reduced two-sided ideal which is a regular ring. If A is left KY-injective, then any reduced principal left ideal is generated by an idempotent (cf. the proof of Proposition 7 (2)). If A is also semi-prime, then any reduced left ideal is a two-sided ideal of A which is a strongly regular ring. (We shall later look into conditions when non-reduced left ideals in semi-prime left KY-injective rings contain central idempotents.)

Remark 6. Let A be semi-prime left KY-injective. Then S, the sum of all reduced left ideals of A, coincides with the sum of all reduced two-sided ideals of A and is the unique maximal strongly regular ideal of A. If, further, every complement left ideal of A is finitely generated, then $A = S \oplus T$, where S is a left and right continuous strongly regular ring and T contains all the nilpotent elements of A.

We are now in a position to mention a few characteristic properties of semi-simple Artinian rings.

Combining [4, Theorem 3.2], Propositions 1 and 7, together with the proof of Theorem 5, we get

Proposition 8. *The following conditions are equivalent:*

(1) A *is semi-simple Artinian;*

(2) *Every cyclic semi-simple* A*-module is* CY*-injective;*

(3) *Every cyclic torsionfree* A*-module is* CY*-injective;*

(4) *Every finitely generated torsionfree* A*-module is* KY*-injective:*

(5) A *is semi-prime left* KY*-injective satisfying the maximum condition on left annihilators.*

Recall that A is directly finite iff ${}_AA \oplus {}_AM \approx {}_AA$ implies M = 0.

Corollary 8.1. *Let* A *be directly finite such that any cyclic torsionfree* A*-module not isomorphic to* ${}_AA$ *is* CY*-injective. Then* A *is either semi-simple Artinian or an integral domain.*

Proof. It is clear that every principal left ideal of A is projective. Suppose that A is not a domain. Then there exists b ∈ A such that l(b) = Ae, where e is a non-trivial idempotent. Since A = Ae ⊕ A(1-e) is directly finite, then both Ae and A(1-e) must be CY-injective and hence injective by Proposition 1. A is therefore left self-injective which implies that every cyclic torsionfree A-module is CY-injective and the corollary follows from Proposition 8.

The next remark also holds.

Remark 7. The following conditions are equivalent: (1) A is either semi-simple Artinian or a left principal ideal domain; (2) A is a directly finite ring such that any left ideal not isomorphic to ${}_AA$ is injective.

We now turn to conditions which will ensure that two injective modules are isomorphic. For any left A-module M, any two-sided ideal T of A, $r_M(T) = \{y \in M / Ty = 0\}$ is a left submodule of M. If M, N are A-modules and f : M → N is a left A-homomorphism, then $f(r_M(J)) \subseteq r_N(J)$.

Theorem 9. *Let* A *be a left* KY*-injective ring satisfying the maximum condition on left annihilators,* M, N A*-modules,* u : M → N, v : N → M *left* A*-monomorphisms. If* E, H *are injective* A*-modules and* f : M → E, g : N → H *are left* A*-monomorphisms such that* $f(r_M(J))$ (resp. $g(r_N(J))$ *is an essential left submodule of* $r_E(J)$ *(resp.* $r_H(J)$*), then* ${}_AE \approx {}_AH$.

Proof. Since ${}_AH$ is injective, there exists a left A-homomorphism $h : E \to H$ such that $gu = hf$. If $y \in \ker h \cap \cap \operatorname{Im} f$, then $y = f(z)$, $z \in M$ and $g(u(z)) = h(f(z)) = h(y) = 0$ implies $u(z) = 0$, whence $z = 0$, yielding $y = 0$. Now $\ker h \cap \cap f(r_M(J)) = 0$ implies $\ker h \cap r_E(J) = 0$ (because $f(r_M(J))$ is essential in $r_E(J)$ (as left A-modules)). First suppose that $J \neq 0$. Since A is left KY-injective satisfying the maximum condition on left annihilators; then by Proposition 7, $J = Z$ is nilpotent. Let n be the least positive integer such that $J^n = 0$. Then $J^n \ker h = 0$ implies $J^{n-1} \ker h \subseteq r_E(J) \cap \cap \ker h = 0$. If $n-1 > 1$, we similarly have $J^{n-2} \ker h = 0$ and so on. Finally, we reach $\ker h = 0$. By symmetry, we also get a left A-monomorphism of H into E. It follows from [3, Theorem 1.13] that ${}_AE \approx {}_AH$. Now if $J = 0$, then A is semi-simple Artinian by Proposition 8 and in that case, ${}_AE \approx {}_AM \approx {}_AN \approx {}_AH$.

The next proposition may be similarly proved.

Proposition 10. (1) *If* A *is semi-primary,* M, N *A-modules such that either* $r_M(Z)$ *is isomorphic to* $r_N(Z)$ *or* $r_M(J)$ *is isomorphic to* $r_M(J)$ *(as left A-modules) then* ${}_AM$ *and* ${}_AN$ *have isomorphic injective hulls;*

(2) *Let* A *satisfy the maximum condition on left annihilators. If* P, Q *are A-modules such that* $r_P(Z)$ *and* $r_Q(Z)$ *are isomorphic (as left A-modules), then* ${}_AP$ *and* ${}_AQ$ *have isomorphic injective hulls.*

Note that, in general, for any non-singular A-module M, $r_M(Z) = M$.

Proposition 11. *Let* A *be a commutative ring,* M *a non-singular injective A-module. For any ideal* T *of* A, $r_M(T)$ *is an injective submodule of* M.

Proof. Let E be an injective hull of $r_M(T)$ in M. For any $y \in E$, there exists an essential ideal L of a such that $Ly \subseteq r_M(T)$ which implies $LTy = TLy = 0$, whence Ty is contained in the singular submodule of M which is zero. Therefore $y \in r_M(T)$ which proves that $r_M(T)$ is an injective A-module.

Corollary 11. *Let* A *be a commutative non-singular ring satisfying the maximum condition on annihilators. For any non-singular divisible* A*-module* M *and any ideal* T *of* A, $r_M(T)$ *is an injective* A*-module (cf. [3,P. 102 ex. 18]).*

The proof of Theorem 9 and Proposition 11 yield

Remark 8. Let A be commutative with a nilpotent ideal U. If M is a submodule of a non-singular A-module N, then M is essential in N iff $r_M(U)$ is essential in $r_N(U)$.

Proposition 12. *The following conditions are equivalent for a commutative ring* A*:*

(1) A *is self-injective regular;*

(2) *For any finitely generated* A*-module* M *and any ideal* P *of* A, $r_{M/Z(M)}(P)$ *is an injective projective* A*-module;*

(3) *For any finitely generated* A*-module* M *and any ideal* P *of* A, $r_{M/Z(M)}(P)$ *is a CY-injective projective* A*-module.*

Proof. Assume (1). For any finitely generated A-module M, we know that N = M/Z(M) is a non-singular A-module which is therefore injective and projective by [9, Corollary 6]. If P is an ideal of A, by Proposition 11, $r_N(P)$ is an injective submodule which is therefore a direct summand of N. Thus (1) implies (2).

(2) implies (3) evidently.

Assume (3). In as much as $_AA/Z$ is projective, we get Z = 0 and hence (3) implies (1) by Propositions 1 and 7.

A theorem of M. Ikeda - T. Nakayama asserts that if A is left self-injective, then for any left ideals L, S of A, $r(L \cap S) = r(L) + r(S)$. We now consider situations where certain proper direct summands of A contain non-trivial central idempotents.

Theorem 13. *Let* A *be a semi-prime left* KY*-injective ring such that* r(L ∩ S) = r(L) + r(S) *for any left ideals* L, S. *Let* I *be a non-singular* CY*-injective left ideal of* A *containing two non-zero principal left ideals* P, Q *with the following properties:* P *contains no direct sum of a pair of mutually isomorphic non-zero left ideals of* A *while* Q *contains no left ideal isomorphic to* P. *If* A = Q⊕K, *then* K *contains a non-trivial central idempotent.*

Proof. For any b ∈ I, there exist a ∈ A such that b = bab (cf. the proof of Proposition 7). Consequently, P and Q are direct summands of $_AA$. By Zorn's Lemma, the set E of all left A-monomorphisms from some submodule of $_AP$ into $_AQ$ contains a maximal member g. Let g : D → Q, D ⊆ P. Since P, Q are also direct summands of I, they are CY-injective a and hence are injective A-modules by Proposition 1. If g(D) = = F, let $\hat{D}$, $\hat{F}$ be the injective hulls of D, F in P, Q respectively. If we suppose that D ≠ $\hat{D}$, then g extends to a left A-homomorphism $\hat{g}$: $\hat{D}$ → Q. Since D is essential in $\hat{D}$, then $\hat{g}$ is a monomorphism belonging to E, which contradicts the maximality of g. This proves that D = $\hat{D}$ and therefore F = $\hat{F}$. If P = D ⊕ Az, z = z^2 ∈ I, then z ≠ 0 (in as much as Q contains no left ideal isomorphic to P). If Q = F ⊕ T, F = Au, T = Av, u, v being idepotents in I, then we claim that zAu = 0. If not, let 0 ≠ w = zdu, d ∈ A. Since l(w) = Ak, k = k^2, the map H : Az → Aw given by H(az) = aw for all a ∈ A yields Az/ker H ≈ Aw. Then ker H = Akz implies Az/Akz ≈ Aw. Since Akz = As, s = s^2 ∈ I, and A = As ⊕ A(1-s), then Az = As ⊕ ⊕ Az(1-s), where Az(1-s) = A(1-s) ∩ Az. Now r = z(1-s) ∈ Az, which implies Ar = Ae, e = e^2 ∈ Az, whence Ae ≈ Ac, where Aw = Ac, c = c^2 ∈ Au, yielding g^{-1}(Ac) ⊕ Ae ⊆ P and g^{-1}(Ac) ≈ ≈ Ae, thus contradicting the hypothesis on P. This proves that zAu = 0. If we suppose that zAv ≠ 0, then there exist similarly non-zero idempotents t ∈ Av, q ∈ Az and an isomorhism m of Aq onto At. Now the map n : D ⊕ Aq → F ⊕ At given by $n(p + aq) = g(p) + m(aq)$ for all $p \in D$, $a \in A$, is a monomorphism which contradicts the maximality of g in E. Therefore $zAv = 0$ also which yields $zQ = 0$, $z \neq 0$. Since Q is a direct summand of ${}_AA$, if $A = Q \oplus K$, then K is the left annihilator of h, where $Q = Ah$, $h = h^2$. If $T = QA$, in as much as A is semi-prime, $l(T) = r(T)$, $l(T) \cap T = 0$ and hence $A = r(l(T)) + r(T) = l(l(T)) + l(T)$ which implies that $A = l(l(T)) \oplus l(T)$. Again, since A is semi-prime, $l(Q) = l(T)$ is generated by a central idempotent and it follows that $K = l(h)$ contains a non-trivial central idempotent of A.

Corollary 13. *Let* A *be a semi-prime self-injective ring with non-zero socle. If* Q *is a non-zero non-singular injective left ideal which contains no minimal left ideal and* $A = Q \oplus V$, *then* V *contains a non-trivial central idempotent.*

We conclude with a last remark.

Remark 9. Let A be a left continuous regular ring, P, Q two non-zero injective left ideals having the same properties as in Theorem 13. If $A = Q \oplus K$, then K contains a non-trivial central idempotent.

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REZIME

O INJEKTIVNIM MODULIMA I UOPŠTENJIMA

U ovom radu dva uopštenja injektivnih modula su uvedena i korišćena za karakterizaciju nekih dobro poznatih klasa prstena.

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