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SOME FURTHER PROPERTIES OF TOTAL AND RICCI CURVATURE IN THE SELF-RECURRENT WEYL-OTSUKI SPACE OF THE SECOND KIND

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ABSTRACT

This note treats some problems concerning the concircularly semi-symmetric metric connection on a Riemannian manifold. Actually, if the main tensor of connection in the Weyl-Otsuki space of the second kind (SW-On) is self-recurrent in the adjoint Riemannian space, the metric covariant connection is semi-symmetric.

1. INTRODUCTION

This research work is mainly based on the results of three relatively fresh papers on the same problem (concircularly semi-symmetric metric connection, π-semi-symmetric metric connection) and in our last paper, published this year, applying the results of [7] on SW-On satisfying $\overset{\circ}{\nabla}_k P^i_j = \pi_k \delta^i_j$, now named the self-recurrent SW-On.

The π-semi-symmetric metric connection is defined in the following way:

$$(1.1)\qquad T(X,Y) = \pi(Y)X - \pi(X)Y$$

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where π is an 1-form, for the torsion tensor

$$\bar{\nabla} g = 0 \tag{1.2}$$

for the metric tensor

$$(\bar{\nabla}_X \pi)(Y) = 0, \tag{1.3}$$

for arbitrary vector fields X,Y.

From (1.1), (1.2), (1.3) this yields

$$\bar{\nabla}_X Y = \nabla_X Y + \pi(Y)X - g(X,Y)P \tag{1.4}$$

where ∇ denotes the Levi-Civita covariant differentiation on the manifold, and P is such a vector field that $g(X,P) = \pi(X)$ for any vector field X.

The same is the meaning of the definition of concircularly semi-symmetric metric connection, but it is expressed in local coordinates (more familiar to us):

$$\Gamma^i_{jk} = \left\{{i \atop jk}\right\} + \delta^i_k p_i - g_{jk} p^i \tag{1.5}$$

$$p_{j,k} = \nabla_k p_j = p_j p_k + \rho g_{jk} \tag{1.6}$$

where $\left\{{i \atop jk}\right\}$ denotes the coefficients of the Levi-Civita connection on the Riemannian manifold, ∇ denotes the covariant differentiation regarding to $\left\{{i \atop jk}\right\}$ and ρ is an arbitrary scalar function.

The main results regarding the concircularly semi-symmetric metric connection (π-semi-symmetric metric connection) from [1], [2], [7] are the following propositions:

Proposition 1. *The concircular curvature tensor of the concircularly semi-symmetric metric connection is equal to the concircular curvature tensor of the Levi-Civita connection.*

Proposition 2. *The projective curvature tensor of the concircularly semi-symmetric metric connection is equal to* *the projective curvature tensor of the Levi-Civita connection.*

Proposition 3. *The conformal curvature tensor of the concirculyrly semi-symmetric metric connection is equal to the conformal curvature tensor of the Levi-Civita connection.*

Proposition 4. *If a Riemannian manifold admits a semi-symmetric metric* π*-connection,* $\bar{\nabla}$*, then the manifold is of constant curvature if and only if it is of constant curvature for* $\bar{\nabla}$.

Proposition 5. *If a Riemannian manifold admits a semi-symmetric metric* π*-connection* $\bar{\nabla}$*, then the manifold is an Einstein manifold if and only if it is an Einstein manifold for the connection* $\bar{\nabla}$.

Proposition 6. *If a Riemannian manifold admits a semi-symmetric metric* π*-connection* $\bar{\nabla}$*, then a necessary and sufficient condition for the Ricci tensor to vanish is either the Levi-Civita curvature tensor being equal to the projective curvature tensor or to the conformal curvature tensor of the Levi--Civita connection.*

Proposition 7. *If a Riemannian manifold admits a semi-symmetric metric connection (*π*-connection) with a covariantly constant torsion tensor, then the manifold is Ricci-symmetric if and only if* $S(X,Y) = -(n-1)\pi(P)g(X,Y)$ $(R_{ij} = -(n-1)\cdot \pi^a\pi_a g_{ij})$.

Proposition 8. *The Ricci tensor of the semi-symmetric metric* π*-connection is symmetric if and only if the 1-form* π *is closed.*

Proposition 9. *If a Riemannian manifold of dimension* n $(n \geq 3)$ *admits a semi-symmetric metric* π*-connection* $\bar{\nabla}$ *with*

P a Killing vector field, then a necessary and sufficient condition for the Ricci-tensor of $\bar{\nabla}$ to be skew-symmetric is that the Ricci tensor of the Levi-Civita connection is given by:

$$S(X,Y) = (n-2)\,\pi(P)g(X,Y) - \pi(X)\pi(Y) \tag{1.7}$$
$$[R_{ij} = (n-2)[\pi^a\pi_a g_{ij} - \pi_i\pi_j]]$$

Proposition 10. *If a Riemannian manifold of dimension* n (n ≥ 3) *admits a semi-symmetric metric* π*-connection whose Ricci tensor is skew symmetric and if* P *is a Killing vector field, then the manifold cannot be Ricci-flat. If* n = = 2 *and* P *is a Killing vector field, the Ricci tensor of* $\bar{\nabla}$ *is skew-symmetric if and only if the manifold is Ricci-flat.*

It has to be noticed that, in a local sense, π^i denote the contravariant components of π, or components of vector P.

We shall apply most of these results to the Weyl-Otsuki space of the second kind (denoted by SW-On). The covariant differentiation in such a space is given by (for example)

$$T^i_{jk,h} = P^i_a\, T^a_{bc|h}\, P^b_j\, P^c_k\ , \tag{1.8}$$

where P^i_j are coefficients of a linear isomorphism of the space, and $T^a_{bc|h}$ is the basic covariant differentiation given by two classical affine connections; one of them ($''\Gamma$) works exceptionally on covariant indices, and the other one ($'\Gamma$) works on contravariant ones. SW-On are metric spaces and there hold the following conditions:

a) $g_{ij,k} = \gamma_k m_{ij}$ (γ_k is a vector field and m_{ij} is a symmetric tensor field)

b) the connection $'\Gamma$ is symmetric

c) $P_{ij} = g_{ia}\, P^a_j$ is symmetric.

The inverse of tensor P is denoted by Q. The underlying Riemannian space is called the adjoint Riemannian space;

its Levi-Civita covariant differentiation is denoted by $\overset{\circ}{\nabla}$. For further properties, the reader is urged to see, for example [3], [4], [5], [6].

If SW-On satisfies the condition

$$(1.9) \qquad \overset{\circ}{\nabla}_k P^i_j = \pi_k \delta^i_j,$$

it is said to be self-recurrent.

2.

Let us consider a self-recurrent SW-On and its connection $\overset{\prime\prime m}{\Gamma}$ (that is, the covariant part of the metric connection if γ_k, from condition a) vanishes). We can calculate its coefficients:

$$(2.1) \qquad \overset{\prime\prime m}{\Gamma}{}^i_{jk} = \{{}^{\,i}_{jk}\} + \tilde{\pi}_j \delta^i_k - g_{jk}\tilde{\pi}^i,$$

where $\tilde{\pi}_j$ means the image of π_j by the linear isomorphism Q. We can see that $\overset{\prime\prime m}{\Gamma}$ is a semi-symmetric metric connection.

In [5] facts were investigated about the conformal curvature tensor of the connection $\overset{\prime\prime m}{\Gamma}$ and about its equality to the conformal curvature tensor of the adjoint Riemannian space. Next, in [6] conditions were investigated for the vector field π_i to satisfy the concircularity condition (and for $\overset{\prime\prime m}{\Gamma}$ to be concircularly semi-symmetric metric). That condition holds in the adjoint Riemannian space for the vector field π_i:

$$(2.2) \qquad \overset{\circ}{\nabla}_k \pi_j = \tilde{\pi}_j \pi_k + \pi_j \tilde{\pi}_k + \rho P_{jk},$$

where ρ is a scalar function. It is easy to show that, if it is properly chosen, π_i is a harmonic vector field. So, for a harmonic vector field π_i, we can apply Propositions 1. and 2. to the self-recurrent SW-On, as we have done in [6].

According to [8], we have:

Corollary 1. *If the one dimensional Betti number of the manifold M is different from zero, the covariant metric connection of the self-recurrent SW-On can be concircularly semi-symmetric metric connection (semi-symmetric metric π-connection)([1]).*

Except for this, two helpful equalities hold:

$$\mathring{\nabla}_k\tilde{\pi}_j = Q^a_j\mathring{\nabla}_k\pi_a - \tilde{\tilde{\pi}}_j\pi_k \tag{2.3}$$
$$\mathring{\nabla}_k Q^i_j = -\pi_k Q^i_a Q^a_j.$$

According to Proposition 4, we have

Corollary 2. *If the one-dimensional Betti number of the manifold M is different from zero, and if π_i is a harmonic vector field, the metric connection* $\overset{m}{\Gamma}$ *is of constant curvature if and only if the adjoint Riemannian space is of constant curvature.*

Suppose that the adjoint Riemannian space is of constant curvature, then

$$R^i_{jk\ell} = k(g_{jk}\delta^i_\ell - g_{j\ell}\delta^i_k) \tag{2.5}$$

$$R_{jk} = (n-1)kg_{jk} \text{ or } R^j_k = (n-1)k\delta^j_k \tag{2.6}$$

$$R = n(n-1)k \tag{2.7}$$

In regard to Corollary 2, analogic relations hold for $\overset{m}{R}{}^i_{jk\ell}$, $\overset{m}{R}_{jk}$, $\overset{m}{R}$, $\overset{m}{K}$.

From [1], we have

$$\overset{m}{R}_{ij} = R_{ij} + (n-1)\tilde{\pi}^a\tilde{\pi}_a g_{ij}, \tag{2.8}$$

and

$$(2.9) \qquad {}^{\prime\prime}\overset{m}{R} = R + n(n-1)\tilde{\pi}^a\tilde{\pi}_a.$$

According to (2.7), we have

$$n(n-1)\,{}^{\prime\prime}\overset{m}{K} = n(n-1)K + n(n-1)\tilde{\pi}^a\tilde{\pi}_a$$

and, consequently

Theorem 1. *If the vector* π_i *is harmonic and if the adjoint Riemannian space is of constant Gaussian curvature* K, *then the connection* ${}^{\prime\prime}\overset{m}{\Gamma}$ *also is of constant Gaussian curvature* ${}^{\prime\prime}\overset{m}{K} = K + \pi^a\pi_a$. *If their total curvatures are equal, the* SW-On *is trivial.*

After this, according to Proposition 5, we have

Proposition 11. *A self-recurrent* SW-On *with* π_i *a harmonic vector field is an Einstein space in regard to linear connection* ${}^{\prime\prime}\overset{m}{\Gamma}$ *if and only if the adjoint Riemannian space is an Einstein space.*

Moreover, taking into account (2.8), we can state

Proposition 12. *The Ricci tensor of* ${}^{\prime}\overset{m}{\Gamma}$ *in a self--recurrent* SW-On *with* π_i *a harmonic vector field is a symmetric tensor if and only if the Ricci tensor of the adjoint Riemannian space is symmetric.*

3.

This section is concerned with the Ricci curvature of a self-recurrent SW-On with harmonic π_i in relation to the Ricci curvature of the adjoint Riemannian space.

Before all the calculations, we can state

Proposition 13. *The Ricci curvature of the adjoint*

Riemannian space cannot be positive definite in the direction π_i.

Now, we shall consider the case when the Ricci tensor of $\overset{m}{\Gamma}''$ vanishes. In this case, we get

$$S(X,Y) = -(n-1)\pi(P)g(X,Y)$$

(3.1)

$$(R_{ij} = -(n-1)\tilde{\pi}^a\tilde{\pi}_a g_{ij})$$

and

(3.2) $$R = -n(n-1)\tilde{\pi}^a\tilde{\pi}_a$$

Consequently, the adjoint Riemannian space is an Einstein space. Then, according to Proposition 6, we have either

(3.3) $$\delta^i_k R_{j\ell} = \delta^i_j R_{k\ell}$$

or

(3.4) $$R^i_k g_{\ell j} - R^i_j g_{\ell k} + R_{\ell j}\delta^i_k - R_{\ell k}\delta^i_j = \frac{R}{n-1}(\delta^i_k g_{\ell j} - \delta^i_j g_{\ell k})$$

So, we have:

Theorem 2. *The necessary and sufficient condition for the Ricci tensor of the connection* $\overset{m}{\Gamma}''$ *to vanish is either (3.3) or (3.4), for the Ricci tensor and the scalar curvature of the adjoint Riemannian space.*

If the Ricci tensor of the connection $\overset{m}{\Gamma}''$ vanishes, so does its scalar curvature. So, we have for this case:

Theorem 3. *The curvature tensor of the connection* $\overset{m}{\Gamma}''$ *is equal to the curvature tensor of the adjoint Riemannian space if and only if either (3.3) or (3.4) holds.*

or

Corollary 3. *The relation*

$$
\begin{aligned}
(3.5)\qquad & (\tilde{\pi}_j\pi_i + \pi_j\tilde{\pi}_i + \rho P_{ij} + \tfrac{1}{2}g_{ij}\tilde{\pi}_a\tilde{\pi}^a)g_{k\ell} - \\
& - (\tilde{\pi}_k\pi_i + \pi_k\tilde{\pi}_i + \rho P_{ki} + \tfrac{1}{2}g_{ki}\tilde{\pi}_a\tilde{\pi}^a)g_{j\ell} - \\
& - (\tilde{\pi}_j\pi_\ell + \pi_j\tilde{\pi}_\ell + \rho P_{j\ell} + \tfrac{1}{2}g_{j\ell}\tilde{\pi}_a\tilde{\pi}^a)g_{ki} + \\
& + (\tilde{\pi}_k\pi_\ell + \pi_k\tilde{\pi}_\ell + \rho P_{k\ell} + \tfrac{1}{2}g_{k\ell}\tilde{\pi}_a\tilde{\pi}^a)g_{ji} = \\
& = \pi_j\pi_i g_{k\ell} - \pi_k\pi_i\, g_{j\ell} - \pi_j\pi_\ell g_{ki} + \pi_k\pi_\ell g_{ij}
\end{aligned}
$$

holds if and only if (3.3) or (3.4) hold.

By the fact of Proposition 8, we have

Corollary 4. *The necessary and sufficient condition for the Ricci tensor of the connection* $''\overset{m}{\Gamma}$ *(metric semi-symmetric* π*-connection) to be symmetric is that the 1-form* $\tilde{\pi}$ *is closed.*

Now, we shall investigate the conditions for the vector $\tilde{\pi}$ to be a Killing vector field. For a Killing vector field, we have

$$
(3.6)\qquad \nabla_j v_i + \nabla_i v_j = 0
$$

and, consequently

$$
(3.7)\qquad \nabla_i v^i = 0
$$

We use relations (2.3) and (3.6) and get

$$
(3.7)\qquad \overset{\circ}{\nabla}_j\tilde{\pi}_i + \overset{\circ}{\nabla}_i\tilde{\pi}_j = \tilde{Q}^a_i\overset{\circ}{\nabla}_j\pi_a - \tilde{\tilde{\pi}}_i\pi_j + Q^a_j\overset{\circ}{\nabla}_i\pi_a - \tilde{\tilde{\pi}}_j\pi_i = 0
$$

which is the Killing equation for the vector $\tilde{\pi}_i = Q^a_i \pi_a$ and the Levi-Civita connection of the adjoint Riemannian space. But, the vector $\tilde{\pi}$ satisfies the concircularity condition. That fact is expressed by (2.2). After that, the Killing equation will take the form

$$\tilde{\pi}_i \tilde{\pi}_j = - \rho g_{ij} \tag{3.8}$$

where ρ is the scalar function appearing in the $\tilde{\pi}$-concircularity condition. We can express (3.8) in the following way:

$$\pi_i \pi_j = - \rho P_{ia} P^a_j = - \rho \overset{2}{P}_{ij} \tag{3.9}$$

(3.8) i.e. (3.9) holds if the vector field $\tilde{\pi}_i$ is a Killing vector field.

Now, if we apply (3.9) to (2.2), the following relation yields

$$\overset{\circ}{\nabla}_k \pi_j = - \rho P_{kj}.$$

As, consequently, we apply successive covariant differentiation to the components of the tensor P_{kj}, we get

$$\overset{\circ}{\nabla}_\ell \overset{\circ}{\nabla}_i P_{kj} = \overset{\circ}{\nabla}_\ell \pi_i g_{kj} = - P_{i\ell} g_{kj} \tag{3.10}$$

and, consequently

$$P_{sj} R^s_{ki\ell} + P_{ks} R^s_{ji\ell} = 0 \tag{3.11}$$

according to the Ricci identity for repeated covariant differentiation, and, hence

$$R^p_{ki\ell} = - Q^{pj} P_{ks} R^s_{ji\ell} \tag{3.12}$$

for the components of the curvature tensor of the adjoint Riemannian space.

So, we have

Theorem 4. *If the vector $\tilde{\pi}_i$ is concircular and Killing simultaneously, the components of the curvature tensor in the adjoint Riemannian space have to satisfy condition (3.12).*

Corollary 5. *If the vector $\tilde{\pi}_i$ is concircular and Killing simultaneously, then*

$$(3.13) \qquad Q^j_p P_{ks} R^s_{ji\ell} = - Q^j_k P_{ps} R^s_{ji\ell}$$

$$(3.13.a) \qquad Q^j_p P_{ks} R^s_{ji\ell} = Q^j_i P_{\ell s} R^s_{jpk}$$

and

$$(3.13.b) \qquad Q_j P_{ks} R^s_{ji\ell} = Q^{\ell j} P_{is} R^s_{jk\ell}$$

Applying the Ricci identity, we can also get the following relation:

$$(3.14) \qquad R^s_{jk\ell}\pi_s = \rho_\ell P_{kj} + \rho\pi_\ell g_{kj} - \rho_k P_{\ell j} - \rho\pi_k g_{\ell j}$$

Now, we are concerned with the possibility of such a realization. We know that a concircularly semi-symmetric metric connection is realizable on a Riemannian manifold if the vector π_i is a harmonic vector field. Then

$$(3.15) \qquad \rho = \frac{-2\tilde{\pi}^k \pi_k}{P^i_i}$$

and

$$0 = - \rho P^i_i = 2\tilde{\pi}^k \pi_k$$

i.e. the vectors π and $\tilde{\pi}$ are mutually orthogonal.

Theorem 5. *The vector $\tilde{\pi}$ can be concircular and Killing simultaneously if its image by the linear isomorphism P^i_j*

is a harmonic vector field and if π and $\tilde{\pi}$ are mutually orthogonal.

Now, we can state first:

Corollary 6. *The Ricci curvature of the adjoint Riemannian space cannot be a negative definite along the direction $\tilde{\pi}_i$, if $\tilde{\pi}$ is orthogonal to π.*

Theorem 6. *For a harmonic vector field π_i and $\tilde{\pi}_i$ orthogonal to it, the Ricci tensor of $\overset{m}{\Gamma}$ is skew-symmetric if and only if the Ricci tensor of the Levi-Civita connection in the adjoint Riemannian space is given by*

$$R_{ij} = (n-2)(\tilde{\pi}^a\tilde{\pi}_a g_{ij} - \tilde{\pi}_i\tilde{\pi}_j) \tag{3.16}$$

or

$$Q^{\ell j}P_{ks}R^s_{ji\ell} = (2-n)(Q^{sa}\pi_a Q^b_s g_{ij} - Q^b_i Q^a_j \pi_b \pi_a). \tag{3.16.a}$$

Finally, we get

Theorem 7. *If the vector π satisfies condition (3.10) and if the Ricci tensor of $\overset{m}{\Gamma}$ is skew-symmetric, then $\overset{m}{\Gamma}$ cannot be Ricci-flat.*

Corollary 7. *If the vector π is harmonic and if $\tilde{\pi}$ is orthogonal to it, then the Ricci curvature of the adjoint Riemanian space is negatively definite along the direction of π and positively definite along the direction of π and positively definite along the direction of $\tilde{\pi}$; its one-dimensional Betti number cannot vanish and, except for this, it admits a one-parametric group of motions. If, moreover, we build a self-recurrent Weyl--Otsuki space of the second kind on such a Riemannian space, the connection $\overset{m}{\Gamma}$ cannot be Ricci flat if its Ricci tensor is skew--symmetric.*

But, if the Ricci tensor of $\overset{m}{T}$ is skew-symmetric, the Gaussian curvature of $\overset{m}{T}$ vanishes. Then, according to Theorem 1. and Theorem 5. we have

Corollary 8. *For a harmonic vector field* π_i *and* $\tilde{\pi}_i$ *orthogonal to it, if the connection* $\overset{m}{T}$ *is of Gaussian curvature zero and if the Ricci tensor of the adjoint Riemannian space is given by the expression*

$$R_{ij} = (n-2)(\tilde{\pi}^a\tilde{\pi}_a g_{ij} - \pi_i\pi_j),$$

then the adjoint Riemannian space has a constant negative total curvature.

REFERENCES

[1] Asoke Kumar Ray, *A note on semi-symmetric -connection on a Riemannian manifold, Indian Journal of Pure and Applied Mathematics 15 (1984), 1206 - 1211.*

[2] B. Barua and Asoke Kumar Ray, *Some properties of a semi-symmetric metric connection in a Riemannian manifold, Indian Journal of Pure and Applied Mathematics 16 (1985), 736 - 740.*

[3] M. Prvanović, *Weyl-Otsuki spaces of the second and third kind, Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. 11, (1981), 219 - 226.*

[4] N. Pušić, *Weyl-Otsuki spaces of the second kind with a special tensor P, Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. 12, (1982), 379 - 386.*

[5] N. Pušić, *A new kind of a metric Otsuki space, Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. 14, (1984), 63 - 71.*

[6] N. Pušić, *On concircular and projective curvature tensors of a certain Weyl-Otsuki space of the second kind, Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. (in print).*

[7] W. Slosarska: *On some invariants of Riemannian manifols admitting a concircularly semi-symmetric metric connection, Demonstratio Mathematica XVII (1984), 251 - 257.*

[8] K. Yano and S. Bochner, *Curvature and Betti numbers*, *Princeton University Press*, 1953.

REZIME

NEKE OSOBINE TOTALNE I RIČIJEVE KRIVINE
SAMOREKURENTNOG VEJL-OTSUKIJEVOG PROSTORA
DRUGE VRSTE

U radu su ispitane osobine totalne i Ričijeve krivine samorekurentnog Vejl-Otsukijevog prostora druge vrste (rad je baziran na rezultatima o koncirkularno-semisimetričnoj metričkoj koneksiji u Rimanovom prostoru. Ispituju se veze samorekurentnog SW-On sa pridruženim Rimanovim prostorom.

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