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THE S-ASYMPTOTIC AND OTHER DEFINITIONS OF THE ASYMPTOTIC BEHAVIOUR OF DISTRIBUTIONS

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ABSTRACT

The paper compares different definitions of the behaviour of a distribution at infinity with the S-asymptotic by some examples and propositions.

INTRODUCTION

In the last thirty years, many definitions of the asymptotic behaviour of distributions have been presented. We can divide them into two sets. To the first one belong those definitions which directly use the classical definition of the asymptotic behaviour of a numerical function. All of them are given only in the one dimensional case. The second set contains definitions which correspond to the distribution a class of distributions depending on some parameters.

Representatives of the first set are definitions given by M.J. Lighthill [5] and by J. Lavoine and O.P. Misra [4].

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Definition 1. (Lighthill's definition). *Distribution* T *behaves as the numerical function* f(x) *at a point* x_0 *if and only if* T *is equal to a numerical function* F *in an open neighbourhood of* x_0 *and* $F(x) \sim f(x),\ x \to x_0$.

The definition used by J. Lavoine and O.P. Misra has been improved by other mathematicians. We shall cite the last we know [8].

Definition 2. (A. Takači's definition). *The distribution* T *is equivalent at infinity with a regularly varying function* $\rho(x) = x^p L(x)$, $p \neq -n$, $n \in N$, *if there exist* $n \in N_0$, $n+p>0$, $a > 1$ *and a continuous function* F *on* R *such that* $T = D^n F(x)$ *on* (a,∞) *and* $F(x) \sim C_{p,n} x^{p+n} L(x)$, $x \to \infty$. *We write then* $T \overset{E}{\sim} \rho(x)$, $x \to \infty$.

L(x) is a slowly varying function, and the constant $C_{p,n}$

$$C_{p,n} = \begin{cases} 1/(p+1) \ldots (p+n), & n \in N \\ 1 & , n = 0. \end{cases}$$

Representatives of the second set are: the quasiasymptotic and S-asymptotic.

The most general definition of the quasiasymptotic is given in book [9]. Before we give this definition, we have to introduce some notations.

Let Γ be a closed, convex, acute and bodied cone in R^n;

$\{U_k,\ k \in I\}$ be a set of linear applications defined in R^n such that $U_k\Gamma = \Gamma$, $k \in I$ and *det* $U_k > 0$, $k \in I$;

$\rho(k)$ be a positive function, $k \in I$;

I be a subset of R having ∞ as a limit point.

Definition 3. (Quasiasymptotic). *Suppose that* $T \in S'$ *with its support in* Γ. T *has the quasiasymptotic in* Γ *related to the set* $\{U_k,\ k \in I\}$ *and the function* $\rho(k)$ *if in* (S')

$$\frac{1}{\rho(k)} T(U_k t) \to g(t),\ k \to \infty,\ k \in I.$$

In our examples we shall use only the one-dimensional case with $\Gamma = [0,\infty)$.

Definition 4. (S-asymptotic). *A distribution* $T \in (D')$ *has the S-asymptotic in cone* Γ*, related to some* $c(h) > 0$, $h \in \Gamma$*, and with the limit* $U \in (D')$ *if there exists*

$$\lim_{h\in\Gamma,\|h\|\to\infty} \langle T(t+h)/c(h),\varphi(t)\rangle = \langle U,\varphi\rangle, \quad \varphi \in (D).$$

Then we write $T(t+h) \overset{S}{\sim} c(h)U(t)$, $h \in \Gamma$, [7].

1. S-ASYMPTOTIC OF A REGULAR DISTRIBUTION AND ASYMPTOTIC BEHAVIOUR OF FUNCTIONS AT INFINITY

According to Lighthill's definition, we have cited, of the behaviour of a distribution at infinity and bearing in mind the local property of the S-asymptotic [7], it is enough to compare the S-asymptotic and asymptotic of numerical functions and we have at the same time a comparison of the S-asymptotic and Lighthill's definition of the behaviour of a distribution at infinity.

The following example (see [1], p. 45) points out that a continuous function can have the S-asymptotic as a distribution without having the asymptotic: Let

$$T(\tau) = \int_{\alpha}^{\tau} g(x)dx, \; g \in L^1(-\infty,\infty) \cap C(-\infty,\infty), \; \alpha > 0.$$

Then

$$T(\tau+\beta) \overset{S}{\sim} 1\cdot \int_{\alpha}^{\infty} g(x)dx, \; \beta \in R_+.$$

By Theorem 1 c) [7] $T^{(1)}(t+\beta) \overset{S}{\sim} 1\cdot(\int_{\alpha}^{\infty} g(x)dx)'$, $\beta \in R_+$. Hence $g(t+\beta) \overset{S}{\sim} 1 \cdot 0$, $\beta \in R_+$.

But g must not have the asymptotic behaviour when $\tau\to\infty$. This example shows that every function from $L^1 \cap C$ has an S-asymptotic zero, when $\beta \to \infty$ or $\beta \to -\infty$, related to $c(h) \equiv 1$.

Let, now, $h(t) = e^t \int_\alpha^t g(x)dx$, where $g(x)$ has the same properties as in the preliminary case. It is easy to see that

$$h(t+\beta) \overset{s}{\sim} e^\beta \cdot e^t \int_\alpha^\infty g(x)dx, \ \beta \in R_+$$

$h'(t) = h(t) + e^t g(t)$ has the S-asymptotic related to e^β, $\beta \in R_+$, just $e^t \int_\alpha^\infty g(x)dx$. But $h'(t)$ must not have an asymptotic related to e^t because of $g(t)$. All derivatives $h^{(k)}(t)$ have the same S-asymptotic as $h(t)$, related to e^β.

The following example shows that a function can have the usual asymptotic behaviour without having an S-asymptotic with the limit U different from zero. This example will be the function: $exp(x^2)$. Suppose that the regular distribution defined by the function $exp(x^2)$ has the S-asymptotic relative to $c(h)>0$, $h \in R_+$ with a limit U different from zero. By Proposition 4 [7], U has the form $U(t) = C\, exp(at)$. Then for every $\varphi \in (D)$, and consequently for $\varphi > 0$ we have

$$\lim_{h\to\infty} \frac{1}{c(h)} \int exp[(x+h+h_0)^2]\, \varphi(x)dx = e^{ah_0} < Ce^{ax}, \varphi(x) >.$$

Therefore

$$e^{ah_0} < U,\varphi > =$$

$$= exp(h_0^2) \lim_{h\to\infty} \frac{1}{c(h)} \int e^{(x+h)^2} e^{2h_0(x+h)} \varphi(x)dx$$

$$\geq exp(h_0^2) < U,\varphi >, \text{ for every } h_0 > 0.$$

But this is not correct.

It is easy to show that for some classes of numerical functions from the asymptotic behaviour at infinity there follows the S-asymptotic. The next proposition is such a one.

P r o p o s i t i o n 1. *Let Γ be a convex cone and $\Omega \subset R^n$*

an open set with the property: for every $r > 0$ *there exists a* β_r *such that* $B(h,r) \subset \Omega$, $h \in \Gamma$, $\|h\| \geq \beta_r$.

Suppose that the function G *is local integrable over* Ω *and has the following properties:*

a) The distribution G_0 *is equal on* Ω *to the distribution* $\tilde{G}_\Omega$ *defined by the function* G.

b) There exist number M_r *and a local integrable function* V *such that*

$$|G(x+h)/c(h)| \leq M_r, \ x \in B(0,r), \ x+h \subset \Omega \ ;$$

$$\lim_{h\in\Gamma,\|h\|\to\infty} G(x+h)/c(h) = V(x), \ x \in B(0,r).$$

Then $G_0(x+h) \overset{S}{\sim} c(h)V(x)$, $h \in \Gamma$.

Proof. Because of the local property of the S-asymptotic [7] it is enough to prove that for a $\varphi \in D$, $supp\varphi \subset B(0,r)$

$$\lim_{h\in\Gamma,\|h\|\to\infty} \int \frac{G(x+h)}{c(h)} \varphi(x)dx = \int V(x)\varphi(x)dx.$$

Bearing in mind the property of Ω and supposition b), we can go to the limit in h under the integral sign.

2. RELATION BETWEEN THE QUASIASYMPTOTIC AND S-ASYPTOTIC

To compare these two notions we have first to fix the basic space and the class of numerical functions c(h). As the quasiasymptotic is defined for tempered distributions (S') and related to regular varying functions ($c(h) = h^a L(h)$, $a \in R$ and L(h) is a slowly varying function) we shall compare, first, in this case, supposing moreover that the limit distribution U differs from zero.

The following example shows the imperfection of the comparison of these two notions. In the following $\theta(t) = 1$, $t > 0$; $\theta(t) = 0$, $t < 0$

i) The regular distribution $\tilde{T} = \theta(t)\, e^{iat}$, $a \neq 0$, has the quasiasymptotic $\frac{i}{a}\delta$ in (S') related to $c(h) = h^{-1}$ [2]:

$$k < \theta(kt)\, e^{ikat}, \varphi(t) > = k \int_0^\infty e^{ikat}\varphi(t)dt$$

$$= \frac{1}{ia} \int_0^\infty \varphi(\tfrac{x}{k})d(e^{iax}) =$$

$$= \frac{1}{ia}\, e^{iax}\varphi(\tfrac{x}{k})\Big|_0^\infty - \frac{1}{k}\int_0^\infty e^{iax}\varphi'(\tfrac{x}{k})dx$$

$$\to \frac{i}{a}\varphi(0), \ k \to \infty.$$

But the distribution $\tilde{T}$ has no S-asymptotic related to a h^α with a $U \neq 0$:

$$< \theta(t+h)e^{ia(t+h)}, \varphi(t) > = e^{iah} \int_{-h}^\infty e^{iat}\varphi(t)dt$$

$$\sim e^{iah}\int_{-\infty}^\infty e^{iat}\varphi(t)dt, \ h \to \infty.$$

This distribution really has an S-asymptotic but relative to the function $c(h) = e^{iah}$.

ii) The regular distribution $T(t) = \theta(t)\sin t$ has quasiasymptotic relative to $c(h) = h^{-1}$ but it has no S-asymptotic at all:

$$h < \theta(ht)\sin ht, \varphi(t) > = \int_0^\infty \sin u\varphi(\tfrac{u}{h})du$$

$$= \varphi(0) + \frac{1}{h}\int_0^\infty \varphi'(\tfrac{u}{h})\cos u \, du.$$

For the S-asymptotic we have

$$< \theta(t+h)\sin(t+h), \varphi(t) > \sim \cos h \int_{-\infty}^{-\infty} \sin t\varphi(t)dt +$$

$$+ \sin h \int_{-\infty}^\infty \cos t\varphi(t)dt, \ h \to \infty.$$

iii) For the regular distribution $\tilde{T} = \theta(t)\sin\sqrt{t}$ we can not find an $\alpha \in R$ and a distribution $U_\alpha \neq 0$ such that

$$\lim_{k\to\infty} \int_0^\infty k^\alpha \sin\sqrt{kt}\ \varphi(t)dt = \langle U_\alpha, \varphi \rangle, \quad \varphi \in (S).$$

Suppose on the contrary that such α and U_α exist. Then we can choose for the φ, $\varphi(t) = e^{-pt}$, $t > 0$, Re $p > 0$. Then we have

$$\lim_{k\to\infty} k^\alpha \int_0^\infty \sin\sqrt{kt}\ e^{-pt}dt = \langle U_\alpha(t), e^{-pt} \rangle.$$

The value of the last integral is $\sqrt{\pi k}/\sqrt{4p^3}\ \exp(-k/4p)$ and $\langle U_\alpha(t), e^{-pt} \rangle$ is the Laplace transform of distribution U_α. In such a way the last relation says that the Laplace transform of U_α equals zero for Re $p > 0$, hence $U_\alpha = 0$.

A proposition which compares these two notions is the following [2]:

Proposition 2. *Suppose that* $f \in (S'_+(R))$ *and* $f(t+h) \overset{S}{\sim} h^\alpha U(t)$, $h \to \infty$, $\alpha > -1$. *Then* f *has a quasiasymptotic of order* α *as well.*

Proof. By relation (6) $\langle f(x+h), \varphi(x) \rangle = (f * \check{\varphi})(h) = H(h)$. This numerical function $H(h)$ has the usual asymptotic of order α, when $h \to \infty$. By Lemma 1 [2], it has the quasiasymptotic at infinity of the same order α. The Fourier transform of distribution f gives $F[f(kx)](p) = \frac{1}{k} F[f](\frac{p}{k})$. From the continuity of the Fourier transform follows that $F[f]$ has the quasiasymptotic at zero of order $\alpha+1$ if and only if f has the quasiasymptotic at infinity of order α.

Suppose now that we chose an $\varphi = \varphi_0 \in (S)$ such that $F[\check{\varphi}](p) = 1$ in a neighbourhood of zero. The Fourier transform of $H(h) = (f * \check{\varphi})(h)$ gives $F[H] = F[f]F[\check{\varphi}]$. We know that $F[H](p)$ has the quasiasymptotic at zero of order $\alpha+1$. With the φ_0 we chose, there follows that $F[f]$ has the quasiasymptotic at zero of order $\alpha+1$ as well. Therefore f has the quasiasymptotic at infinity of the order α.

If we change the basic space, the conclusion can be quite different. Suppose that the basic space is (K_1) (see [6])

Elements of (K_1) are functions φ from C^∞ such that

$$\nu_k(\varphi) = \sup_{x\in R, n<k} e^{k|x|}|D^n\varphi| < \infty .$$

The function $\theta(x)\,Ch x = \frac{1}{2}\theta(x)(e^x + e^{-x})$ defines an element from (K_1') and $exp(-px^2)$, $p > 0$, is from (K_1).

To find the quasiasymptotic at infinity in (K_1') of $\theta(x)Ch(x)$ we shall analyse the integral

$$\int_0^\infty Ch(kx)e^{-px^2}dx = \frac{1}{2}\int_0^\infty Ch(k\sqrt{x})e^{-px}\frac{dx}{\sqrt{x}} = \frac{\sqrt{\pi}}{\sqrt{4p}}exp(k^2/4p).$$

This shows that there exists no function $c(k)$ of the form $e^{ak^r}k^bL(k)$, $a,b \in R$, $0 \le r \le 2$, $L(k)$ slowly varying function, such that $\theta(x)Chx$ has the quasiasymptotic relative to $c(k)$ with a limit $U \neq 0$.

For the S-asymptotic we have for $\varphi \in (K_1)$:

$$\lim_{h\to\infty} e^{-h}\int_{-h}^\infty C(x+h)\varphi(x)dx =$$

$$= \frac{1}{2}\int_{-h}^\infty e^x\varphi(x)dx + \frac{1}{2}e^{-2h}\int_{-h}^\infty e^{-x}\varphi(x)dx$$

$$\to \frac{1}{2}\int_{-\infty}^\infty e^x\varphi(x)dx,\ h\to\infty.$$

Therefore $\theta(x)Chx$ has the S-asymptotic relative to $c(h) = e^h$ with the limit $U = \frac{1}{2}e^x$.

3. EQUIVALENCE AT INFINITY OF A DISTRIBUTION AND THE S-ASYMPTOTIC

By Definition 2, the equivalence at infinity of a distribution is related only to the power growth such as $x^pL(x)$ of a distribution. Therefore, the comparison has a meaning only for such distributions.

If we have regular distributions, the difference between these two definitions of asymptotic behaviour of distributions follows from the fact that the S-asymptotic preserves the usual asymptotic of numerical functions (Theorem 1.b) [7]), but the equivalence at infinity generalizes l'Hospital's rule with Stolz's improvement. Namely, this rule says: Suppose that the real valued function H(x) has the first derivative for $x > x_0$; $H'(x) \neq 0$, $x > x_0$ and $H(x) \to \infty$, $x \to \infty$; the function F(x) has its first derivative. If there exists

$$\lim_{x\to\infty} F'(x)/H'(x) = A, \text{ then there exists}$$

$$\lim_{x\to\infty} F(x)/H(x) = A.$$

We know that the opposite does not hold. Definition 2 uses just the opposite statement in l'Hospital's rule, with $H(x) = x^{p+1}$. The next example shows this disagreement with the usual asymptotic by numerical functions:

$$F(x) = \frac{1}{2}\, x^{-\frac{1}{2}} + e^x \cos e^x = D(x^{\frac{1}{2}} + \sin e^x).$$

By Definition 2, the distribution defined by function F(x) is equivalent at infinity with $\frac{1}{2}\, x^{-\frac{1}{2}}$.

By reason of the same fact, the function

$$F(x) = x^{-\frac{1}{2}} + \sin x = D(2x^{\frac{1}{2}} - \cos x)$$

is equivalent at infinity with $x^{-\frac{1}{2}}$ but has no S-asymptotic with a limit $U \neq 0$ (see 2 . ii)).

Without any difficulty one can find a function which has the S-asymptotic but for which Definition 2 does not hold. We shall give such a one, not trivial:

$$F(x) = e^{-x}\sin(\frac{\pi}{2} - e^{-x}) = D(\sin(\frac{\pi}{2} - e^{-x})).$$

For $x > 0$

$$1 \geq \sin(\frac{\pi}{2} - e^{-x}) \geq \sin(\frac{\pi}{2} - 1).$$

Hence

$$\frac{x^{n-1}}{(n-1)!} \geq \underbrace{\int_0^x \int_0^y \dots \int_0^t}_{n-1} \sin\left(\frac{\pi}{2} - e^{-u}\right) du \dots dy$$

$$\geq \frac{x^{n-1}}{(n-1)!} \sin\left(\frac{\pi}{2} - 1\right), \ n \in N.$$

The function $F(x)$ can be written in the form $F(x) = D^n E(x)$, where $E(x)$ is

$$E(x) = \underbrace{\int_0^x \int_0^y \dots \int_0^t}_{n-1} \sin\left(\frac{\pi}{2} - e^{-u}\right) du \dots dy.$$

From the previous inequality it follows that the behaviour at infinity could be x^{n-1}, but Definition 2 does not admit such a power.

Proposition 3. *Suppose that* T *is equivalent at infinity with* $Ax^p L(x)$, *then there exists* n_0 *such that* T *has the S-asymptotic related to* $c(h) = h^{p+n} L(h)$, $n \geq n_0$ *and with the limit* $U = 0$.

Proof. By Definition 2, there exists a continuous function $f(x)$, $x \in R$ such that $T = D^k f(x)$, $x > x_0$ and

$$f(x) \sim Ax^{p+k}/(p+1) \dots (p+k) L_1(x), \ x \to \infty.$$

Function $f(x)$ has the property that for a $C \in R^+$ and every compact set $K \subset R$

$$\left|\frac{f(x+h)}{h^{p+k} L(h)}\right| = \left|\frac{f(x+h)}{(x+h)^{p+k} L(h)}\right| \frac{|(x+h)^{p+k}|}{h^{p+k}} \leq C, \ x \in K, \ h \geq h_K$$

and

$$\lim_{h\to\infty} f(x+h)/h^{p+k} L(h) = A/(p+1) \dots (p+k).$$

By Theorem 1.b) [7] $f(x+h) \overset{s}{\sim} h^{p+k} L(h) A/(p+1) \dots (p+k)$, $h \to \infty$, h > 0. Now from Theorem 1.c) [7] follows the statement of our proposition with $n_0 = k$.

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REZIME

S-ASIMPTOTIKA I DRUGE DEFINICIJE ASIMPTOTSKOG PONAŠANJA DISTRIBUCIJE

U poslednjih trideset godina matematička literatura je obogaćena sa više definicija ponašanja distribucija u „beskonačnosti". Autori tih definicija polazili su ili od izučavanja odredjenih matematičkih modela, pre svega iz fizike, ili od mogućnosti matematičkog aparata. Uporedjivanje tih definicija otežava činjenica da su one po svojoj strukturi različite. Tako, dok jedne odražavaju lokalni karakter distribucije u okolini „beskonačno udaljene tačke", druge definicije vezane su za globalni karakter distribucije. I pored tih teškoća, na raznim primerima, kontraprimerima i tvrdjenjima ukazano je na odnose različitih definicija ponašanja distribucija u „beskonačnosti" i S-asimptotike na čijoj teoriji je i sam autor radio.

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