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SOME CONGRUENCES ON A STRONGLY π-INVERSE r-SEMIGROUP

*Petar Protić** and *Stojan Bogdanović***

* *Gradjevinski fakultet Niš, 18000 Niš, Jugoslavija*

** *Institut za matematiku Novi Sad, Dr. Ilije Djuričića 4, 21000 Novi Sad, Jugoslavija*

ABSTRACT

In this paper we describe a congruence pair for a strongly π-inverse r-semigroup and in this way we obtain a generalization of a result of Petrich [5].

A semigroup S is *π-regular* if for every $a \in S$ there exists a positive integer m such that $a^m \in a^m S a^m$. We shall denote by RegS the set of all regular elements of S. An element a' is an inverse for a if $a = aa'a$ and $a' = a'aa'$. As usual, we shall denote by V(a) the set of all inverses of a. If A is a subset of S, then $V(a) = \bigcup_{a \in A} V(a)$. A semigroup S is *π-orthodox* if S is π-regular, and the set E(S) of all idempotents of S is a subsemigroup of S [1]. A semigroup S is *strongly π-inverse* if it is π-regular and the idempotents commute [1].

1. THE KERNEL OF A CONGRUENCE ON A π-ORTHODOX SEMIGROUP

Let S be a π-regular semigroup. Let a mapping $r : S \to$ RegS be defined by $r(a) = a^m$, where m is the smallest posi-

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tive integer such that $a^m \in \mathrm{Reg}S$. The mapping r is a surjection.

Definition 1.1. *A relation* ρ *on a* π*-regular semigroup* S *is* r*-semiprime if*

$$(\forall a \in S)\ a\ \rho\ r(a).$$

Lemma 1.1. *If* ρ *is an* r*-semiprime congruence on a* π*-regular semigroup* S*, then* S/ρ *is a regular semigroup and*

$$V(a\rho) = V(r(a)\rho).$$

Proof. Let $a \in S/\rho$. Then $a\ \rho\ r(a)$ and there exists $x \in S$ such that $r(a) = r(a)xr(a)$. So $a\rho = r(a)\rho = (r(a)xr(a))\rho = r(a)\rho x\rho r(a)\rho = a\rho x\rho a\rho$. Thus S/ρ is regular.

If A is a subset of a π-regular semigroup S, then

$$\mathrm{reg}A = \{a \in A : a \in \mathrm{Reg}S\} = A \cap \mathrm{Reg}S$$

and

$$\mathrm{Reg}A = \{a \in A : (\exists x \in A)\ a = axa\}.$$

A subsemigroup K of a semigroup S is *full* if $E(S) \subseteq K$.

Definition 1.2. *Let* S *be a* π*-regular semigroup. A subset* A *of* S *is self-conjugate if* $a'(\mathrm{reg}A)a \subseteq \mathrm{reg}A$ *for every* $a \in \mathrm{Reg}S$ *and* $a' \in V(a)$*. A subsemigroup* K *of* S *is inverse-closed if* $V(\mathrm{reg}K) \subseteq \mathrm{reg}K$, K *is strongly inverse-closed if* $V(\mathrm{Reg}K) \subseteq \mathrm{Reg}K$.

Definition 1.3. *A subsemigroup* K *of a* π*-orthodox semigroup* S *is normal if it is full, self-conjugate and inverse-closed.*

Definition 1.4. *A subsemigroup* K *of a* π*-regular semigroup* S *is* r*-semiprime if*

$$(\forall a \in S)(r(a) \in K \Rightarrow a \in K).$$

Let ρ be a congruence on a semigroup S. The restriction $\rho \mid E(S)$ is the *trace* of ρ to be denoted by $\mathrm{tr}\rho$, and the set

$$\ker\rho = \{a \in S : (\exists e \in E(S))\ a\ \rho\ e\}$$

is the *kernel* of ρ [5].

Lemma 1.2. *If* S *is a* π-*orthodox semigroup, then*

$$a'ea,\ aea' \in E(S)$$

for every $e \in E(S)$, $a \in \mathrm{Reg}S$, $a' \in V(a)$.

Proof. By Proposition VI 1.4 [2].

Theorem 1.1. *Let* ρ *be a (r-semiprime) congruence on a* π-*othodox semigroup* S. *Then* $\ker\rho$ *is a normal (r-semiprime) subsemigroup of* S.

Proof. It is clear that $\ker\rho$ is full. Let $a \in \mathrm{Reg}S$ and $a' \in V(a)$. Then

$$a'(\mathrm{reg}(\ker\rho))a = \bigcup \{a'ba : b \in \mathrm{reg}(\ker\rho)\}.$$

Since $b \in \mathrm{reg}(\ker\rho)$ we have that $b\ \rho\ e$ for some $e \in E(S)$. So $a'ba\rho a'ea$. Now by Lemma 1.2 we obtain that $a'ba \in \ker\rho$. Since S is π-orthodox we have by Proposition IV 3.1. [4] that RegS is a subsemigroup of S. So $a'ba \in \mathrm{reg}(\ker\rho)$. Hence $\ker\rho$ is a full, self-conjugate subsemigroup of S. Furthermore, RegS is an orthodox semigroup and

$$\ker_{|\mathrm{Reg}S} = \ker\rho \cap \mathrm{Reg}S = \mathrm{reg}(\ker\rho).$$

From this and Lemma 2.3 [3], we obtain

$$V(\ker\rho_{|RegS}) \subseteq \ker\rho_{|RegS}$$

i.e.

$$V(reg(\ker\rho)) \subseteq reg(\ker\rho).$$

Hence, $\ker\rho$ is normal.

If ρ is an r-semiprime congruence and $a \in S$, $r(a) \in \ker\rho$, then there exists $e \in E(S)$ such that $a\rho r(a)\rho e$ whence $a \in \ker\rho$. Thus ker is an r-semiprime subsemigroup of S.

Proposition 1.1. *If* K *is a full* π*-regular subsemigroup of a* ρ*-regular semigroup* S *and* RegS *is a subsemigroup of* S, *then* K *is a strongly inverse-closed subsemigroup of* S.

Proof. Let $a \in RegK$, $a' \in V(a)$ such that $a' \in RegK$ and let $x \in V(a)$. Then

$$x = xax = x(aa'a)x = (xa)a'(ax) \in E(S)\cdot RegK\cdot E(S) \subseteq$$
$$\subseteq RegK.$$

Thus $V(a) \subseteq RegK$.

Proposition 1.2. *If* K *is an inverse-closed subsemigroup of a* π*-regular semigroup* S, *then* regK = RegK *and* K *is a* π*-regular subsemigroup of* S.

Proof. Clearly $RegK \subseteq regK$. Let $a \in regK$ and $x \in V(a)$ Since $V(regK) \subseteq regK$ we have that $V(a) \subseteq regK \subseteq K$. So $x \in K$. Hence, $a \in RegK$. Thus $regK \subseteq RegK$.

Corollary 1.1. *Let* ρ *be a (r-semiprime) congruence on a* π*-orthodox semigroup* S. *Then* $\ker\rho$ *is a normal* π*-orthodox (r-semiprime) subsemigroup of* S *and* $reg(\ker\rho) = Reg(\ker\rho)$.

Proof. Follows immediately by Theorem 1.1 and Proposition 1.2.

2. CONGRUENCE PAIR

Now we shall introduce the following notion.

Definition 2.1. *A π-regular semigroup is an r-semigroup if*

$$(\forall a,b \in S)(r(ab) = r(a)r(b)).$$

Every regular semigroup is an r-semigroup, but the converse is not true. For example, the semigroup S given by the table

	0	e	a	b
0	0	0	0	0
e	0	e	a	b
a	0	0	0	0
b	0	b	a	e

is a strongly π-inverse r-semigroup and it is not regular.

Lemma 2.1. *Let* S *be an r-semigroup. Then* RegS *is a subsemigroup of* S *and*

$$r : S \to \mathrm{Reg}S$$

is a homomorphism.

Proof. *Let* a,b ∈ RegS. Then

$$ab = r(a)r(b) = r(ab) \in \mathrm{Reg}S.$$

Definition 2.2. *Let* S *be a strongly π-inverse semigroup. A congruence* ξ *on the semilattice* E(S) *is normal if*

$$e\xi f \Leftrightarrow a^{-1}ea\xi a^{-1}fa$$

for $e,f \in E(S)$, $a \in \mathrm{Reg}S$ *and* a^{-1} *inverse of* a.

Definition 2.3. *Let* S *be a strongly* π*-inverse semigroup and let* K *be a normal r-semiprime subsemigroup of* S. *If, in addition,* ξ *is a normal congruence on* $E(S)$ *satisfying*

(i) $r(a)e \in K,\ e\xi r(a)^{-1}r(a) \Rightarrow r(a) \in K$,

(ii) $a \in K \Rightarrow r(a)^{-1}er(a)\xi r(a)^{-1}r(a)e$

for every $e \in E(S)$, $a \in S$, *then* (ξ,K) *is a congruence pair for* S.

Define a relation $K_{(\xi,K)}$ on S by

$$aK_{(\xi,K)}b \Leftrightarrow r(a)^{-1}r(a)\xi r(b)^{-1}r(b),\ r(a)r(b)^{-1} \in K.$$

It is clear that $K_{(\xi,K)}$ is an r-semiprime relation.

Lemma 2.2. *Let* (ξ,K) *be a congruence pair for a strongly* π*-inverse r-semigroups. Then*

(i) $r(a)er(b) \in K,\ e\xi r(a)^{-1}r(a) \Rightarrow r(ab) \in K$,

(ii) $ab \in K \Rightarrow r(a)er(b) \in K$,

(iii) $r(a)r(b)^{-1} \in K,\ r(a)^{-1}r(a)\xi r(b)^{-1}r(b) \Rightarrow$

$\Rightarrow r(a)^{-1}er(a)\xi\ er(b)^{-1}er(b)$

for every $e \in E(S)$, $a,b \in S$.

Proof. (i) Let $r(a)er(b) \in K$ and $e\xi r(a)^{-1}r(a)$. Since S is r-semiprime, we have

(1) $r(a)er(b) = r(a)e(r(b)r(b)^{-1})r(b) =$

$$= r(a)(r(b)r(b)^{-1})er(b) = (r(a)r(b))(r(b)^{-1}er(b)) =$$

$$= r(ab)(r(b)^{-1}er(b)) \in K.$$

Since ξ is a normal congruence on $E(S)$, we have

(2) $$r(ab)^{-1}r(ab) = (r(a)r(b))^{-1}r(a)r(b) =$$

$$= r(b)^{-1}r(a)^{-1}r(a)r(b)\xi r(b)^{-1}er(b).$$

Now, by (1) and (2) and by Definition 2.3 (i), we obtain that $r(ab) \in K$, since $r(b)^{-1}er(b) \in K$.

(ii) Let $a,b \in S$, $ab \in K$ and $e \in E(S)$. Then

$$r(a)er(b) = r(ab)(r(b)^{-1}er(b)) \in K \cdot E(S) \subseteq K,$$

since $E(S) \subseteq K$.

(iii) Let $r(a)r(b)^{-1} \in K$, $r(a)^{-1}r(a)\xi r(b)^{-1}r(b)$ and $e \in E(S)$. Then by Definition 2.3 (ii), we have that

$$r(b)r(a)^{-1}\, er(a)r(b)^{-1}\xi(r(b)r(a)^{-1})(r(a)r(b)^{-1})e$$

i.e.

$$r(b)(r(a)^{-1}er(a))r(b)^{-1}\xi r(b)(r(a)^{-1}r(a))r(b)^{-1}e$$
$$\xi r(b)(r(b)^{-1}r(b))r(b)^{-1}e = r(b)r(b)^{-1}e.$$

Now we have

$$r(a)^{-1}er(a) = (r(a)^{-1}r(a))(r(a)^{-1}er(a))(r(a)^{-1}r(a))$$
$$\xi(r(b)^{-1}r(b))(r(a)^{-1}er(a))(r(b)^{-1}r(b)$$

$$= r(b)^{-1}(r(b)r(a)^{-1}er(a)r(b)^{-1})r(b)$$
$$\xi r(b)^{-1}(r(b)r(b)^{-1}e)r(b) = r(b)^{-1}er(b).$$

Theorem 2.1. *Let (ξ,K) be a congruence pair for a strongly π-inverse r-semigroup. Then $K_{(\xi,K)}$ is an r-semiprime congruence on S with the trace ξ and the kernel K. Conversely, if ρ is an r-semiprime congruence on a strongly π-inverse r-semigroup, then $(\mathrm{tr}\rho, \ker\rho)$ is a congruence pair for S and* $\rho = K_{(\mathrm{tr}\rho,\ker\rho)}$.

Proof. Let (ξ,K) be a congruence pair for a strongly π-inverse r-semigroup and let $\mathcal{K} = K_{(\xi,K)}$. Since $E(S) \subseteq K$, we have that $\mathcal{K}$ is reflexive. Furthermore, since K is inverse-closed, we have that $\mathcal{K}$ is symmetric. That $\mathcal{K}$ is transitive follows by Lemma 2.2.(i). Hence $\mathcal{K}$ is an equivalence.

Assume that $a\mathcal{K}b$ and $c \in S$. Then

$$r(ac)^{-1}r(ac) = (r(a)r(c))^{-1}r(a)r(c) =$$

$$= r(c)^{-1}r(a)^{-1}r(a)r(c)\xi r(c)^{-1}r(b)^{-1}r(b)r(c) =$$

$$= (r(b)r(c))^{-1}r(b)r(c) = r(bc)^{-1}r(bc).$$

Now, since ξ is a normal convergence on $E(S)$, we have by Lemma 2.2. (ii) that $r(a)r(b)^{-1} \in K$ whence

$$r(a)(r(c)r(c)^{-1})r(b)^{-1} = r(a)r(c)(r(b)r(c))^{-1} =$$

$$= r(ac)r(bc)^{-1} \in K.$$

Thus $ac\mathcal{K}bc$. Furthermore, by Lemma 2.2.(iii), we have that $a\mathcal{K}b$ imply $r(a)^{-1}er(a)\xi r(b)^{-1}er(b)$ so

$$r(ca)^{-1}r(ca) = (r(c)r(a))^{-1}r(c)r(a) =$$

$$= r(a)^{-1}r(c)^{-1}r(c)r(a)\ \xi r(b)^{-1}r(c)^{-1}r(c)r(b) =$$

$$= r(cb)^{-1}r(cb).$$

Now, since K is self-conjugate, we have that

$$r(ca)r(cb)^{-1} = r(c)r(a)r(b)^{-1}r(c)^{-1} \in r(c)\,K\,r(c)^{-1} \subseteq K$$

thus $caKcb$. Therefore, K is a congruence. It is clear that K is r-semiprime and that $\mathrm{tr}K = \xi$. Let $a \in \ker K$. Then there exists $e \in E(S)$ such that

$$aKe \Leftrightarrow r(a)^{-1}r(a)\xi e,\ r(a)e \in K,$$

whence (by Definition 2.3 (i)) $(r(a) \in K$. So $a \in K$. Conversely, let $a \in K$. Then $r(a) = r(a)r(a)^{-1}r(a) \in K$ and $r(a)^{-1}r(a)\cdot \xi r(a)^{-1}r(a) = r(a)^{-1}r(a)r(a)^{-1}r(a)$. Since $r(a)^{-1}r(a) \in E(S)$, we have that $aKr(a)^{-1}r(a)$. Thus $a \in \ker K$. Therefore, $K = \ker K$.

Conversely, let ρ be an r-semiprime congruence on a strongly π-inverse r-semigroup S. By Corollary 1.1, $\ker\rho$ is a normal strongly π-inverse r-semiprime subsemigroup of S. Since $\mathrm{tr}\rho$ is a normal congruence on $E(S)$, it remains to prove that conditions (i) and (ii) from Definition 2.2 hold. Assume that $r(a)c \in \ker\rho$ and $e(\mathrm{tr}\rho)r(a)^{-1}r(a)$, (where $a \in S, e \in E(S)$). Then there exists $f \in E(S)$ such that $r(a)e\rho f$. Thus $r(a) \in \ker\rho$, and therefore condition (i) holds. Furthermore, if $a \in \ker\rho$, then $r(a) \in \ker\rho$. So $r(a)\rho e$ for some $e \in E(S)$. From this it follows that $r(a)^{-1}r(a)\rho r(a)^{-1}e$. Now, since $e\rho r(a)$, we have that $r(a)^{-1}r(a)e\rho r(a)^{-1}er(a)$, i.e. (ii) holds.

Now, assume that $a\,K_{(\mathrm{tr}\rho,\ker\rho)}\,b$. Then $r(a)r(b)^{-1}\rho e$ for some $e \in E(S)$. Since $r(a)r(b)^{-1}$ and $r(b)r(a)^{-1}$ are mutually inverse, we have that $r(b)r(a)^{-1} \in \ker\rho$ and there exists $f \in E(S)$ such that $r(b)r(a)^{-1}\rho f$. Now

$$r(a)r(b)^{-1} = r(a)r(b)^{-1}r(b)r(a)^{-1}r(a)r(b)^{-1}\rho efe = ef.$$

So $ef\rho r(a)r(b)^{-1}\rho e$. Similarly, $r(b)r(a)^{-1}\rho fef = ef\rho f$. Thus $e\rho f$ and $r(b)r(a)^{-1}\rho e$. From this it follows that

$$r(a) = r(a)(r(a)^{-1}r(a))\rho r(a)(r(b)^{-1}r(b)) =$$

$$= (r(a)r(b)^{-1})r(b)\rho er(b)$$

and

$$r(b) = r(b)(r(b)^{-1}r(b))\rho r(b)(r(a)^{-1}r(a)) =$$

$$= (r(b)r(a)^{-1})r(a)\rho er(a).$$

So

$$r(a)\rho er(b)\rho e(er(a)) = er(a)\rho r(b)$$

and since ρ is r-semiprime, we have that $a\rho b$. Thus $K_{(tr\rho,ker\rho)} \subseteq \rho$. Conversely, let $a\rho b$. Then $r(a)\rho r(b)$, whence $r(a)r(b)^{-1}\rho r(b)r(b)^{-1}$. So $r(a)r(b)^{-1} \in ker\rho$. Also

(3) $r(a)^{-1}r(a)\rho r(a)^{-1}r(b)$ and $r(b)^{-1}r(a)\rho r(b)^{-1}r(b)$.

Hence $r(a)^{-1}r(b), r(b)^{-1}r(a) \in ker\rho$. Now we have that $r(a)^{-1}r(b)\rho r(b)^{-1}r(a)\rho e$ for some $e \in E(S)$ and by (3) we obtain that $r(a)^{-1}r(a)\rho r(b)^{-1}r(b)$. Thus $aK_{(tr\rho,ker\rho)}b$, i.e. $\rho \subseteq K_{(tr\rho,ker\rho)}$. Therefore, $\rho = K_{(tr\rho,ker\rho)}$.

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REZIME

NEKE KONGRUENCIJE NA STROGO π-INVERZNOJ POLUGRUPI

U ovom radu opisujemo kongruencijski par na strogo π-inverznoj polugrupi S, pri čemu uopštavamo poznati rezultat M. Petricha [5].

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