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THE ADJOINT OPERATOR AND K-CONVERGENCE[1)]

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ABSTRACT

The continuity of an adjoint operator for a linear operator defined on a normed K-space is proved. As a consequence Banach closed graph theorem is obtained.

1. INTRODUCTION

The notion of a K-space is a good replacement for the complete space. Namely, at the present time there are a lot of theorems of functional analysis in which the notion of complete space is replaced by K-space (see for example [4, 5 and 6]).

In this paper we shall prove the continuity of the adjoint operator for a linear operator defined on a normed K-space. Then we obtain Banach closed graph and open mapping theorem as consequences.

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Let (X,τ) be a topological group. A sequence $\{x_n\}$ from X is a *K-sequence* if each subsequence of $\{x_n\}$ has a subsequence $\{y_n\}$ such that

$$\left\{ \sum_{k=1}^{n} y_k \right\}$$

converges to an element $x \in X$.

A topological group which has the property that any sequence which converges to 0 is a K-sequence is called a *K-space*.

The preceding definitions of K-sequence and K-space also hold in the presented forms for sequentially convergent group ([2], [6]).

2. THE ADJOINT OPERATOR

Let X and Y be normed linear spaces and let T be a linear operator with a domain $D(T)$ which is a dense subspace of X, and with the values in Y. We define the adjoint operator T^* for the operator T in the following way: the domain $D(T^*)$ of T^* is

$$D(T^*) = \{y^* | y^* \in Y^*, y^*T \text{ is continuous on } D(T)\}$$

and $T^* : D(T^*) \to X^*$ is defined by $T^*(y^*)$ is the unique continuous extension of y^*T to X. Then, we have

$$T^*(y^*)(x) = y^*(T(x))$$

$(x \in D(T),\ y^* \in D(T^*))$.

Now we have the main statement.

Theorem 1. *Let* X *and* Y *be normed spaces. Let* $T : X \to Y$ *be a linear operator. If* X *is a K-spece, then* T^* *is a linear bounded operator on* $D(T^*)$.

Proof. If $D(T^*) = \{0\}$, then the theorem is trivially true. Suppose that $D(T^*) \neq \{0\}$. Let $\{y_n^*\}$ be an arbitrary sequence from $D(T^*)$ such that $\|y_n^*\| \leq 1$. We shall prove that the sequence $\{T^*(y_n^*)\}$ is bounded which will imply the conclusion of the theorem.

We choose a sequence $\{x_n\}$ from X such that $\|x_n\| = 1$ and

$$\|T^*(y_n^*)\| \leq |T^*(y_n^*)(x_n)| + 1 \quad (n \in N). \tag{1}$$

Let $\{\alpha_n\}$ be an arbitrary sequence of scalars such that $\alpha_n \to 0$. It is obvious that there exist two sequences $\{t_n\}$ and $\{u_n\}$ such that $\alpha_n = t_n u_n$, $t_n \geq 0$, $t_n \to 0$ and $u_n \to 0$ for $n \to \infty$.

We define

$$x_{ij} = t_i |(T^*(y_i^*)(u_j x_j)|$$

for $i \neq j$ and $x_{ii} = 0$ $(i \in \mathbb{N})$. $u_j x_j \to 0$ for $j \to \infty$ implies $x_{ij} \to 0$ for $j \to \infty$ $(i \in \mathbb{N})$. We have to prove $x_{ij} \to 0$ for $i \to \infty$ $(j \in \mathbb{N})$. Namely, we have

$$x_{ij} = t_i |T^*(y_i^*)(u_j x_j)| = t_i |y_i^*(T(u_j x_j))| \leq$$

$$\leq t_i \|T(u_j x_j)\|.$$

Letting $i \to \infty$, we obtain $x_{ij} \to 0$ for each fixed $j \in \mathbb{N}$.

Then by the Diagonal Theorem from [1], we obtain an increasing sequence $\{p_n\}$ of natural numbers such that

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} x_{p_i p_j} = 0. \tag{2}$$

Since X is a K-space and $u_j x_j \to 0$ for $j \to \infty$ there exists a subsequence $\{s_j\}$ of $\{p_j\}$ and an element $x \in X$ such that

$$\sum_{j=1}^{\infty} u_{s_j} x_{s_j} = x.$$

We have for each $p \in \mathbb{N}$

$$t_{s_i} |T^*(y^*_{s_i})(u_{s_i} x_{s_i})| \leq \sum_{\substack{j=1 \\ i \neq j}}^{i+p} t_{s_i} |T^*(y^*_{s_i})(u_{s_j} x_{s_j})| +$$

$$+ t_{s_i} \left|T^*(y^*_{s_i}) \left(\sum_{j=1}^{i+p} u_{s_j} x_{s_j} \right)\right| \qquad (i \in \mathbb{N}).$$

Letting $p \to \infty$ we obtain

$$t_{s_i} |T^*(y^*_{s_i})(u_{s_i} x_{s_i})| \leq \sum_{j=1}^{\infty} x_{s_i s_j} + t_{s_i} |T^*(y^*_{s_i})(x)|.$$

Letting $i \to \infty$, we obtain by (2) from the preceding inequality

$$\alpha_{s_i} |T^*(y^*_{s_i})(x_{s_i})| \to 0.$$

Since the scalars satisfy the Urysohn property: if each subsequence $\{v_n\}$ of a sequence of scalars $\{r_n\}$ has a subsequence $\{z_n\}$ such that $z_n \to 0$, then $r_n \to 0$, we conclude that

$$\alpha_n |T^*(y^*_n)(x_n)| \to 0 \text{ for } n \to \infty.$$

Hence by (1) we obtain

$$\alpha_n \| T^*(y^*_n) \| \to 0 \text{ for } n \to \infty.$$

That implies the boundedness of the adjoint operator T^*.

Remark. Theorem 1. is a generalization of Theorem 2 from [4] when we restrict this theorem to linear mappings.

In the proof of the following corollary we shall need the notion of a weak continuous operator. A linear operator $T : X \to Y$ is weakly continuous if it is continuous with respect to the topologics $\sigma(X,X^*)$ and $\sigma(Y,Y^*)$.

Corollary. **(Banach Closed Graph Theorem).** *Let X and Y be Banach spaces. If $T : X \to Y$ is a linear closed operator, then T is bounded.*

Proof. Since T is a closed operator, the domain $D(T^*)$ is weak * dense in Y^*. Hence T is weakly continuous. Namely, if

$$U = \{y \mid |y_i^*(y)| < \varepsilon_j, \quad j = 1,\dots,n\}$$

for $\varepsilon_j > 0$ and $y_i^* \in Y^*$ is a neighborhood of zero in Y with respect to $\sigma(Y,Y^*)$, then, since $y^*(T(x)) = T^*(y^*)(x) \quad (x \in X)$, i.e. $T^*(y^*) \in X^*$,

$$T^{-1}(U) = \{x \mid |y^*(T(x))| < \varepsilon_j, \ j = 1,\dots,n\}$$

is a neighborhood of zero in X with respect to $\sigma(X,X^*)$. This implies the weakly continuity of operator T. Hence, and by theorem from [5] (let X be a normed K-space and let Y be a normed space. If $T : X \to Y$ is a linear weakly continuous operator, then T is continuous.), we obtain the conclusion of the theorem.

Let us notice that the Closed Graph theorem implies the Open mapping theorem for the Banach space case (see for example [3], (a) Theorem 1.1.3 $\Rightarrow$ Theorem 1.2.4, p. 30).

Postscript. Theorem 1. was cited in the monograph „P. Antosik, C. Swartz, Matrix Methods in Analysis, Lecture Notes in Mathematics, Vol. 1113, 1985". Instead of the Diagonal Theorem from [1] in the proof the Basic Matrix Theorem 2. was used.

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REZIME

ADJUNGOVANI OPERATOR I K-KONVERGENCIJA

Za linearni operator definisan nad normiranim K--prostorom a sa vrednostima u normiranom vektorskom prostoru uvodi se adjungovani operator i dokazuje se da je on uvek ograničen operator - Teorema 1. Na osnovu ove teoreme se dobija jednostavan dokaz teoreme o zatvorenom grafiku za Banahove prostore.

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