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PHILLIPS LEMMA FOR SEMIGROUP VALUED ADDITIVE SET FUNCTIONS

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ABSTRACT

In this paper two generalizations of Phillips lemma for additive set functions with values in a commutative semigroup endowed with special functionals are proved.

1. INTRODUCTION

Phillips lemma [8] is a well-known result in measure theory and has many useful applications in measure theory and functional analysis ([1], [2], [8], [11]). We shall give two generalizations of this lemma for semigroup valued additive set functions. Our methods are based on versions of Drewnowski lemma and on Diagonal theorem.

2. EXSHAUSTIVE ADDITIVE SEMIGROUP VALUED SET FUNCTIONS

Let X be a uniform complete commutative semigroup with a neutral element. Let Σ be a σ-algebra of subsets of a nonempty

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set.

A set function $\mu:\Sigma \to X$ is said to be additive if for any two disjoint sets E_1 and E_2 from Σ

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2).$$

Further we suppose that there exists $B \in \Sigma$ such that $\mu(B) = 0$. Hence $\mu(\emptyset) = 0$, since we have

$$0 = \mu(B \cup \emptyset) = \mu(B) + \mu(\emptyset) = \mu(\emptyset).$$

By H. Weber [10], there exists on X a family of pseudometrics $\{d_i\}_{i\in I}$ such that they generate the uniformity on X and which satisfies

$$(d_+) \quad d_i(x_1 + x_2, y_1 + y_2) \leq d_i(x_1, y_1) + d_i(x_2, y_2)$$

$$(x_1, x_2, y_1, y_2 \in X;\ i \in I).$$

So further on for simplicity we restrict ourselfs on the case of a commutative semigroup endowed with a pseudometric d which satisfies (d_+).
This pseudometric d defines a triangular functional ([5, 6, 7]) $f(x) := d(x,0)$, which has the following properties:

$$(F_1) \quad f(x + y) \leq f(x) + f(y),$$

$$(F_2) \quad f(x + y) \geq f(x) - f(y),$$

$$(F_3) \quad f(0) = 0.$$

We remark that if X is a commutative group, then a triangular functional reduces on a quasi-norm [5]. Theorem (1.1) from [10] implies: There exists on uniform semigroup X a family F of triangular functionals such that a sequence (x_n) from X converges to 0 iff $f(x_n) \to 0$ for each $f \in F$.

Further on we restrict ourselves on one triangular functional f. So we can introduce the notion of exhaustive (s-bounded) set functions in the usual way:

A set function $\mu:\Sigma \to X$ is exhaustive if for each disjoint sequence (E_n) from Σ

$$\lim_{n\to\infty} f(\mu(E_n)) = 0.$$

A *semivariation* of an additive set function $\mu: \Sigma \to X$ is a set function $\hat{\mu}$ defined by

$$\hat{\mu}(E) := \sup \{f(\mu(H)) | H \in \Sigma,\ H \subset E\} \quad (E \in \Sigma).$$

Semivariation $\hat{\mu}$ has the following properties ([6]):

1) $\hat{\mu}(\emptyset) = 0$;
2) $f(\mu(E)) \leq \hat{\mu}(E) \quad (E \in \Sigma)$;
3) $\hat{\mu}(F) \leq \hat{\mu}(E)$ for $F \subset E$, $F,E \in \Sigma$;
4) $\hat{\mu}(E_1 \cup E_2) \leq \hat{\mu}(E_1) + \hat{\mu}(E_2)$ for $E_1 \cap E_2 = \emptyset$, $E_1, E_2 \in \Sigma$
5) μ is exhaustive iff $\hat{\mu}$ is exhaustive.

Now we give a generalization of L. Drewnowski lemma [4]:

THEOREM 1. *Let* X *be a commutative semigroup endowed with a pseudometric* d *which satisfies* (d_+) *and let* (μ_n), $\mu_n:\Sigma\to X$ $(n \in N)$, *be a sequence of additive and exhaustive set functions. Then each disjoint sequence* (E_n) *of sets from* Σ *contains a subsequence* (E_{k_n}) *such that* μ_n, $n \in N$, *are countable additive on the* σ*-algebra generated by* (E_{k_n}).

PROOF. First, we take only one additive exhaustive set function $\mu: \Sigma \to X$. Since the semivariation $\hat{\mu}$ is also exhaustive - property 5) and subadditive - property 4) we can apply Lemma from [4], p. 727. By if there exists a subsequence (E_{k_n}) of the sequence (E_n) such that for each sequence (B_n) of sets from

σ-algebra Σ_0 generated by the subsequence (E_{k_n}), such that $B_{n+1} \subset B_n$ and $\bigcap_n^{\infty} B_n = \phi$, we have $\hat{\mu}(B_n) \to 0$ as $n \to \infty$.

Let (A_n) be a disjoint sequence from Σ_0. Then we have by (d_+)

$$(1) \qquad d\left(\sum_{k=1}^{n} \mu(A_k), \mu\left(\bigcup_{k=1}^{\infty} A_k\right)\right) \leq$$

$$\leq d\left(\sum_{k=1}^{n} \mu(A), \mu\left(\bigcup_{k=1}^{n} A_k\right)\right) + d\left(0, \mu\left(\bigcup_{k=n+1}^{\infty} A_k\right)\right).$$

By the additivity of μ the first sum on the right part of the preceding inequality is zero. Since the inequality

$$d\left(0,\mu\left(\bigcup_{k=n+1}^{\infty} A_k\right)\right) \leq \hat{\mu}\left(\bigcup_{k=n+1}^{\infty} A_k\right)$$

implies

$$d\left(0, \mu\left(\bigcup_{k=n+1}^{\infty} A_k\right)\right) \to 0 \quad \text{as } n \to \infty,$$

we obtain by (1) the σ-additivity of the set function μ.

Now, let (μ_n), $\mu_n: \Sigma \to X$, be a sequence of exhaustive and additive set functions. First , we introduce the space $X^{\mathbf{N}}$ of all X-valued sequences. We define a pseudometric on $X^{\mathbf{N}}$ in the following way

$$D(x,y): = \sum_{k=1}^{\infty} \frac{d(x_k,y_k)}{2^k(1 + d(x_k,y_k))},$$

where $x = (x_k)$ and $y = (y_k)$ are from $X^{\mathbf{N}}$.

Obviously pseudometric D satisfies also the condition (d_+). Now,we introduce a set function $\mu: \Sigma \to X^{\mathbf{N}}$ by

$$\mu(A): = (\mu_1(A),\mu_2(A),\dots)(A \in \Sigma).$$

Obviously μ is additive and exhaustive. Then by the first part of the proof for each disjoint sequence (E_n) there exists a subsequence (E_{k_n}) such that μ is countable additive on the σ-algebre Σ_0 generated by (E_{k_n}). This implies the σ-additivity of each set function μ_n, $n \in \mathbb{N}$, on the σ-algebre Σ_0.

Now, we shall give another generalization of Drewnowski's lemma to semigroup valued set functions. Namely, we take directly a commutative semigroup with a triangular functional f.

In this case the pseudometric $|f(x) - f(y)|$ $(x,y \in X)$ does not satisfy (d_+) in general (for example take the set of all integers Z with the usual addition and $f(x) = |x|$ $(x \in Z)$). Namely, in this case the operation is not continuous in general - [5]. Instead of countable we introduce f-countable additive set function $\mu:\Sigma\to X$, i.e. for each disjoint sequence (E_n) from Σ

$$\lim_{n\to\infty} f(\sum_{i=1}^{n} \mu(E_i)) = f(\mu(\bigcup_{i=1}^{\infty} E_i)).$$

THEOREM 2. *Let X be a commutative semigroup endowed with a triangular functional f and let (μ_n), $\mu_n: \Sigma \to X$ $(n \in \mathbb{N})$, be a sequence of additive and exhaustive set functions. Then each disjoint sequence (E_n) of set from Σ contains a subsequence (E_{k_n}) such that μ_n, $n\in\mathbb{N}$, are f-countable additive on the σ-ring generated by (E_{k_n}).*

The proof is analogous to the proof of the preceding Theorem 1. Instead of (1) we use the inequality

$$|f(\sum_{k=1}^{n} \mu(A_k)) - f(\mu(\bigcup_{k=1}^{\infty} A_k))| \leq f(\mu(\bigcup_{k=n+1}^{\infty} A_k))$$

(Lemma from [5]) and instead of pseudometric D we take triangular functional F defined by

$$F(x) = \sum_{k=1}^{\infty} \frac{f(x_k)}{2^k(1 + f(x_k))}.$$

3. GENERALIZATIONS OF PHILLIPS LEMMA

Now, we give two generalizations of Phillips lemma to semigroup valued additive set functions.

THEOREM 3. *Let* X *be a commutative semigroup with a triangular functional* f *and neutral element* 0 *and let* (μ_n), $\mu_n : \Sigma \to X$ $(n \in N)$, *be a sequence of exhaustive and additive set functions. If*

(2) $$\lim_{n\to\infty} f(\mu_n(E)) = 0 \quad (E \in \Sigma),$$

then for each disjoint sequence (E_j) *from* Σ

(3) $$\lim_{i\to\infty} f(\sum_{j\in A} \mu_i(E_j)) = 0$$

for each $A \subset N$, *where for infinite* A

$$f(\sum_{j\in A} \mu_i(E_j)) := \lim_{n\to\infty} f(\sum_{s=1}^{n} \mu_i(E_{j_s})) \quad (i \in N)$$

and (j_s) *is the increasing sequence of all elements from the set* A.

REMARK. It may happen that there does not exist an element $x_i \in X$ such that

$$f(x_i) = f(\sum_{j\in A} \mu_i(E_j)).$$

We need in the proof the following

ANTOSIK-MIKUSIŃSKI-PAP DIAGONAL THEOREM [5]. *Let* X *be a commutative semigroup with a neutral element* 0 *endowed with a triangular functional* f. *If* $x_{ij} \in X$, $(i,j \in N)$ *and*

$$\lim_{j\to\infty} f(x_{ij}) = 0 \quad (i \in N),$$

then there exist an infinite set I *of natural numbers and a sub-*

set J *(finite or infinite) of* I *such that, for all* $i \in I$, *we have*

$$\sum_{j\in J} f(x_{ij}) < \infty$$

and

$$f(\sum_{j\in J} x_{ij}) \geq \frac{1}{2} f(x_{ii}),$$

where for infinite J

$$f(\sum_{j\in J} x_{ij}) := \lim_{n\to\infty} f(\sum_{s=1}^{n} x_{ij_s})$$

for the increasing sequence (j_s) *of all elements from* J.

PROOF OF THEOREM 3. Suppose that the theorem is not true, i.e. there exists a disjoint sequence (E_j) from Σ such that

$$\lim_{i\to\infty} f(\sum_{j\in N} \mu_i(E_j)) \neq 0.$$

Then there exist $\varepsilon > 0$ and an increasing sequence (p_n) of natural numbers such that

(4) $$f(\sum_{j\in N} \mu_{p_n}(E_j)) > \varepsilon \quad (n \in N).$$

This implies the existence of on index $n_1 \in N$ such that

$$f(\sum_{j=1}^{n_1} \mu_{p_1}(E_j)) > \varepsilon .$$

Now, by (2) there exists $k_1 \in N$ such that

(5) $$\sum_{j=1}^{n} f(\mu_{p_i}(E_j)) < \frac{\varepsilon}{2} \quad \text{for } p_i \geq k_1 .$$

Again by (4) there exists $n_2 > n_1$ such that

(6) $$f\Big(\sum_{j=1}^{n_2} \mu_{k_1}(E_j)\Big) > \varepsilon .$$

Using the property (F_1) of triangular functional f we have

$$f\Big(\sum_{j=1}^{n_2} \mu_{k_1}(E_j)\Big) \leq f\Big(\sum_{j=1}^{n_1} \mu_{k_1}(E_j)\Big) + f\Big(\sum_{j=n_1+1}^{n_2} \mu_{k_1}(E_j)\Big).$$

By the preceding inequality, (5) and (6) we obtain

$$f\Big(\sum_{j=n_1+1}^{n_2} \mu_{k_1}(E_j)\Big) > \frac{\varepsilon}{2} .$$

Repeating the preceding procedure we obtain two increasing sequences (n_i) and (k_i) of natural numbers such that

$$f\Big(\sum_{j=n_i+1}^{n_{i+1}} \mu_{k_i}(E_j)\Big) > \frac{\varepsilon}{2} \quad (i \in N).$$

Then we have the disjoint sequence (A_i),

$$A_i := \bigcup_{j=n_i+1}^{n_{i+1}} E_j ,$$

for which

(7) $$f(\mu_{k_i}(A_i)) > \frac{\varepsilon}{2} \quad (i \in N)$$

holds.

By Theorem 2 there exists a subsequence (A_{n_i}) of (A_i) such that μ_{k_i} are f-countable additive on the σ-algebra generated by (A_{n_i}). Let $x_{ij} := \mu_{k_i}(A_{n_j})$ $(i,j \in N)$. Since μ_{k_i} $(i \in N)$ are exhaustive we have

$$\lim_{j\to\infty} f(x_{ij})=0 \quad \text{and}$$

we can apply on $[x_{ij}]$ $(i,j \in N)$ Antosik-Mikusiński-Pap Diagonal

Theorem. So there exists an infinite subset I of N and its subset J such that

$$f\Big(\sum_{j\in J} \mu_{k_i}(A_{n_j})\Big) \geq \frac{1}{2} f(\mu_{k_i}(A_{n_i})) \quad (i \in I).$$

Let

$$B := \bigcup_{j\in J} A_{n_j}.$$

Then by the preceding inequality and (7) we have

$$f(\mu_{k_i}(B)) > \frac{\varepsilon}{4} \qquad (i \in I).$$

A contradiction to (2) .

THEOREM 4. *Let* X *be a sequentially complete commutative uniform semigroup with neutral element* 0 *and let* (μ_n), $\mu_n : \Sigma \to X$, $(n \in \mathbb{N})$ *be a sequence of exhaustive and additive set functions. If*

(8) $$\lim_{n\to\infty} \mu_n(E) = 0$$

exists for each $E \in \Sigma$, *then for each disjoint sequence* (E_j) *from* Σ

(9) $$\lim_{i\to\infty} \sum_{j\in A} \mu_i(E_j) = 0$$

for each $A \subset \mathbb{N}$.

PROOF. Let d be a pseudometric from the family of pseudometrics which generates the uniformity on X and which satisfies the condition (d_+). Let $f(x) := d(x,0)$ $(x \in X)$. Then by the proof of preceding theorem using now Theorem 1 instead of Theorem 2 we obtain the assertion.

REMARK. If we take in the preceding theorem

$$\lim_{i\to\infty} d(\mu_i(E),\mu(E)) = 0 \quad \text{for each } d\in\{d_i\}_{i\in I}$$

and each $E \in \Sigma$ instead of (8), then we obtain instead of (9)

$$\lim_{i\to\infty} d\Big(\sum_{j\in A} \mu_i(E_j), \sum_{j\in A} \mu(E_j)\Big) = 0$$

for each $A \subset N$ and each $d \in \{d_i\}_{i\in I}$.

We obtain Theorem 8 from [1] as a consequence of Theorem 4.

COROLLARY 1. *Let* G *be a sequentially complete commutative group endowed with a quasi-norm and let* (μ_n), $\mu_n : \Sigma \to G$ $(n \in N)$, *be a sequence of strongly additive set functions. If*

$$\lim_{i\to\infty} \mu_i(E) = \mu(E)$$

exists for each E, *then for each disjoint sequence* (E_j) *from* Σ

$$\lim_{i\to\infty} \sum_{j\in A} \mu_i(E_j) = \sum_{j\in A} \mu(E_j)$$

uniformly for $A \subset N$.

PROOF. The assertion follows by Remark to Theorem 4 and Theorem 3 from [1].

COROLLARY 2 (PHILLIPS LEMMA). *Let* P(N) *be the power set of* N *and let* (μ_n), $\mu_n : P(N) \to R$ $(n \in N)$, *be a sequence of bounded and additive set functions. If*

$$\lim_{n\to\infty} \mu_n(E) = 0 \qquad (E \in N),$$

then

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} |\mu_i(\{j\})| = 0.$$

PROOF. We obtain by Theorem 4 for arbitrary $\varepsilon > 0$

$$\sum_{j\in A} |\mu_i(\{j\})| < \frac{\varepsilon}{2} \quad \text{for } A \subset N$$

and i large. Then we have

$$\sum_{j=1}^{\infty} |\mu_i(\{j\})| \leq \varepsilon$$

for i large [9, 1.1.2].

REFERENCES

[1] P. Antosik, C. Swartz, *The Schur and Phillips Lemmas for Topological Groups, J. Math. Anal. Appl., 98 (1984), 179 - 187.*

[2] P. Antosik, C. Swartz, *Matrix Methods in Analysis, Lect. Notes in Math. 1113, Springer - Verlag, 1985.*

[3] L. Drewnowski, *Topological rings of sets, continuous set functions, integration I,II,III, Bull. Acad. Polon. Sci., 20 (1972), 259 - 279, 610 - 628.*

[4] L. Drewnowski, *Equivalence of Brooks-Jewett, Vitali-Hahn-Saks and Nikodym Theorems, Bull. Acad. Polon. Sci., 20 (1972), 725 - 731.*

[5] E. Pap, *A generalization of the Diagonal theorem on a block-matrix, Mat. ves., 11(26) (1974), 66 - 71.*

[6] E. Pap, *Funkcionalna analiza, Institute of Mathematics, Novi Sad, 1982.*

[7] E. Pap, *Triangular functionals on commutative semigroups, Convergence Struct. and Appl. II, Akademie -*

Verlag, Berlin, 1984, 167 - 172.

[8] R. Phillips, *On Linear Transformations*, Trans. Amer. Math. Soc., 48(1940), 516 - 541.

[9] A. Pietsch, *Nukleare Lokalkonvexe Räume*, Akademie - Verlag, Berlin, 1965.

[10] H. Weber, *Fortsetzung von Massen mit Werten in uniformen Halbgruppen*, Arch. Math. 27 (1976), 412 - 423.

[11] A. Wilansky, *Modern Methods in Topological Vector Spaces*, McGraw-Hill, 1978.

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REZIME

FILIPSOVA LEMA ZA ADITIVNE SKUPOVNE FUNKCIJE SA VREDNOSTIMA U POLUGRUPI

U radu su dokazana dva uopštenja Filipsove leme. Σ je σ-algebra skupova.

TEOREMA 3. Neka je X komutativna polugrupa sa trougaonom funkcionelom f i neutralnim elementom o i neka je (μ_n), $\mu_n : \Sigma \to X$ $(n \in N)$, niz ekshaustivnih i aditivnih skupovnih funkcija. Ako je

$$\lim_{n\to\infty} f(\mu_n(E)) = 0 \quad (E \in \Sigma),$$

tada za svaki niz (E_j) disjunktnih skupova iz Σ važi

$$\lim_{i\to\infty} f(\sum_{j\in A} \mu_i(E_j)) = 0$$

za svaki skup $A \subset N$, gde je za beskonačan skup A

$$f(\sum_{j\in A} \mu_i(E_j)) := \lim_{n\to\infty} f(\sum_{s=1}^{n} \mu_i(E_{j_s}))$$

$(i \in N)$ i (j_s) je rastući niz svih elemenata skupa A.

Drugo uopštenje - Teorema 4., se odnosi na aditivne i ekshaustivne skupovne funkcije sa vrednostima u sekvencijalno kompletnoj komutativnoj uniformnoj polugrupi sa neutralnim elementom.