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THE APPROXIMATE SOLUTION OF A CLASS OF DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper a particular solution of a linear partial differential equation with constant coefficients, in the field of Mikusiński operators is analysed. The approximate solution and the error of the approximation is constructed.

We are going to observe the differential equation:

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \alpha_{\mu,\nu} s^{\nu} x^{(\mu)}(\lambda) = f(\lambda) \tag{1}$$

with conditions:

$$x(0) = \varphi_0 , \; x'(0) = \varphi_1, \ldots, x^{(m-1)}(0) = \varphi_{m-1} \tag{2}$$

in the field of Mikusinski operators, where $x(\lambda)$ is the unknown operator function, $\alpha_{\mu,\nu}$ are numerical constants, $f(\lambda)$ is the continuous operator function and φ_i, $i = 1,2,\ldots,m-1$ are given operators.

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The solution of equation (1) is $x(\lambda) = x_h(\lambda) + x_p(\lambda)$ where the homogeneous part has the form ([1]):

$$(3)\qquad x_h(\lambda) = \sum_{j=1}^{m} b_j \exp(\lambda\omega_j) \quad \text{where} \quad \omega_j = \sum_{i=0}^{\infty} c_{i,j}\, \ell^{\frac{i-p}{q}}$$

b_j are operators determined from (2), $c_{0,j} \neq 0$, $j = 1,2,\dots,m$; ω_j are the solutions of the characteristic polynomial:

$$(4)\qquad P(\omega) = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \alpha_{\mu,\nu}\, s^{\nu}\omega^{\mu} .$$

In paper [1] the approximate solution of homogeneous part $x_h(\lambda)$ was constructed:

$$(5)\qquad \tilde{x}_h(\lambda) = \sum_{j=1}^{m} b_j \exp(\lambda\tilde{\omega}_j) \quad \text{where} \quad \tilde{\omega}_j = \sum_{i=0}^{i_0} c_{i,j}\, \ell^{\frac{i-p}{q}}$$

The approximation of the particular solution was given in [4] for special conditions (2) (namely, for $\varphi_i = 0$, $i = 0,1,\dots,m-1$ and $\varphi_{m-1} = \ell$).

In this paper we shall analyse the particular solution, construct the approximate one and find the error of estimation, supposing that operators $\varphi_0,\dots,\varphi_{m-2}$ are different from zero.

As it is shown in [3] the polynomial of form (4) can always be decomposed into linear factors:

$$P(\omega) = a_m \prod_{j=1}^{m} (\omega - \omega_j), \qquad a_m = \sum_{\nu=0}^{n} \alpha_{\mu,\nu} s^{\nu}$$

Putting $P(\omega) = a_m P_1(\omega)$ one can find constant operators $\bar{A}_j$, $j = 1,2,\dots,m$ such that:

$$\frac{I}{P(\omega)} = \frac{I}{a_m}\,\frac{I}{P_1(\omega)} = \frac{I}{a_m}\left(\frac{\bar{A}_1}{\omega-\omega_1} + \frac{\bar{A}_2}{\omega-\omega_2} + \dots + \frac{\bar{A}_m}{\omega-\omega_m}\right)$$

provided that ω_j, $j = 1,2,\dots,m$ are the logaritmic and simple solutions of (4). Then the solution of equation (1) with conditions (2) is $x(\lambda) = x_h(\lambda) + x_p(\lambda)$. It can be shown that the particular solution has the form:

(6) $$x_p(\lambda) = \sum_{j=1}^{m} A_j \int_0^{\lambda} \exp((\lambda - \kappa)\omega_j) f(\kappa) d\kappa$$

where

(7) $$A_j = \frac{\bar{A}_j}{a_m} = \frac{I}{a_m} \frac{I}{P_1'(\omega_j)} = \frac{I}{P'(\omega_j)}$$

and satisfies the following conditions: $x_p(0) = 0$, $x_p'(0) = 0, \ldots$ $\ldots$, $x_p^{(m-1)}(0) = 0$.

Coefficients A_j. In this part we shall consider the coefficients A_j and construct them approximately. Let us suppose that there exist only one pair of numbers μ_1 and ν_1 such that $1 \leqslant \mu_1 \leqslant m$ and $0 \leqslant \nu_1 \leqslant n$ and

(8) $$\max_{\substack{0\leqslant\mu\leqslant m\\ 0\leqslant\nu\leqslant n}} (\nu + \frac{p}{q}(\mu-1)) = \nu_1 + \frac{p}{q}(\mu_1-1); \quad p \leqslant 1,\ q \in N$$

Then, introducing $\bar{W}_j$ as $\bar{W}_j = \ell^{p/q}\omega_j$ we have:

$$P'(\omega_j) = \mu_1 s^{\nu_1+\frac{p}{q}(\mu_1-1)} \alpha_{\mu_1,\nu_1}(\bar{W}_j^{\mu_1-1} +$$
$$+ \sum_{\mu=1}^{m}{}^{*} \sum_{\nu=0}^{n}{}^{*} \frac{\mu\alpha_{\mu,\nu}}{\mu_1\alpha_{\mu_1,\nu_1}} \ell^{\nu_1-\nu+\frac{p}{q}(\mu_1-\mu)} \bar{W}_j^{\mu-1})$$

where the stars in the sums mean that only the pair of indexes μ_1 and ν_1 together are omited.

LEMMA 1: *If* $\mu = 2,\ldots,m$, $c_{o,j} \neq 0$, $j = 1,\ldots,m$ *and* q *is a natural number, then* $\bar{W}_j^{\mu-1}$ *can be written as:*

$$\bar{W}_j^{\mu-1} = c_\mu I + \ell^{1/q} W_{j,\mu},$$

where c_μ *is a numerical constant and* $\ell^{1/q} W_{j,\mu}$ *represent integrable functions on the interval* $[0,T]$ *for* $T > 0$ *and continuous functions for* $t > 0$.

PROOF: Indeed, $\bar{W}_j$ has the form:

$$\bar{W}_j = C_{0,j} I + \ell^{1/q} W' \qquad c_{1,j} \neq 0$$

and therefore

$$(9) \qquad \bar{W}_j^{\mu-1} = c_{0,j}^{\mu-1} I + \mu c_{0,j}^{\mu-2} I \ell^{1/q} W' + \ldots (\ell^{1/q} W')^{\mu-1} =$$

$$= c_\mu I + \ell^{1/q} W_{j,\mu}$$

so we can conclude that c_μ is a numerical constant and $\ell^{1/q} W_{j,\mu}$ represents an integrable function on the interval $[0,T]$ for $T > 0$ and a continuous function for $t > 0$.

LEMMA 2: *If we denote*

$$(10) \qquad B_j = \sum_{\mu=1}^{m}{}^{*} \sum_{\nu=0}^{n}{}^{*} \frac{\mu \alpha_{\mu,\nu}}{\mu_1 \alpha_{\mu_1,\nu_1}} \ell^{\nu_1 - \nu + p/q(\mu_1 - \mu)} \bar{W}_j^{\mu-1}, \quad j = 1,\ldots,m$$

then B_j *represents an integrable function on the interval* $[0,T]$ *for* $T > 0$ *and a continuous one for* $t > 0$.

PROOF: From (8) follows that $\nu_1 - \nu + p/q(\mu_1 - \mu) = b > 0$ for $\mu \neq \mu_1$ and $\nu \neq \nu_1$ and $W_j^{\mu-1}$ has form (9) so B_j represents a function with the properties mentioned in the Lemma.

Using (9) and (10) $P'(\omega_j)$ can be transformed as:

$$P'(\omega_j) = \mu_1 \alpha_{\mu_1,\nu_1} s^{\nu_1 + p/q(\mu_1 - 1)} (c_{\mu_1} I + \ell^{1/q} W_{j,\mu_1} + B_j)$$

and therefore the coefficient A_j can be written in the form:

$$(11) \qquad A_j = \frac{I}{P'(\omega_j)} = \frac{\ell^{\nu_1} + p/q(\mu_1 - 1)}{\mu_1 \alpha_{\mu_1,\nu_1} c_{\mu_1}} \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{c_{\mu_1}^k} (\ell^{1/q} W_{j,\mu_1} + B_j)^k$$

PROPOSITION: *The series determined in (11) converges absolutely in* $\mathbf{M}$.

PROOF: It is known [3] that the series $\sum_{i=1}^{n} a_i \ell^{i\alpha}$ converges absolutely for $\alpha > 0$ (a_i are numerical coefficients). From Lemma 1 and Lemma 2 it follows that the sum $\ell^{1/q} W_{j,\mu_1} + B_j$

are operators representing integrable functions on interval $[0,T]$, $T > 0$ and continuous for $t > 0$.

Let us, now suppose that there exists more then one pair of numbers $\mu_{1,i}$, $\nu_{1,i}$, $i = 1,\dots,r$, $r < m$ such that:

$$\max_{\substack{1\leq\nu\leq n\\ 1\leq\mu\leq m}} (\nu + p/q(\mu-1)) = \nu_{1,1} + p/q(\mu_{1,1}-1) =$$

$$= \nu_{1,2} + p/q(\mu_{1,2}-1) = \dots = \nu_{1,r} + p/q(\mu_{1,r}-1).$$

Then we may transform $P'(\omega)$ similarly as we have just done. We remark that the analysis of coefficients A_j is essentially the same but technically more complicated. Therefore, further, we shall continue the consideration when there exists only one pair of numbers μ_1 and ν_1 with the mentioned properties.

The approximate solution. In order to approximate the particular solution let us form the approximation of coefficients A_j:

$$\tilde{A}_j = \frac{\ell^{\nu_1+p/q(\mu_1-1)}}{\mu_1\alpha_{\mu_1,\nu_1}\, c_{\mu_1}} \cdot \sum_{k=0}^{k_0} \frac{(-1)^k}{c^k} (\tilde{W}_{j,\mu_1} + \tilde{B}_j)^k \tag{12}$$

where

$$\tilde{B}_j = \sum_{\mu=1}^{m}{}^{*} \sum_{\nu=0}^{n}{}^{*} \frac{\mu\alpha_{\mu,\nu}}{\mu_1\alpha_{\mu_1,\nu_1}} \ell^{\nu_1-\nu+p/q(\mu_1-\mu)} (c_\mu I + \tilde{W}_{j,\mu})^{\mu-1} \tag{13}$$

$$\tilde{W}_{j,\mu} = \sum_{r=1}^{\mu-1} \binom{\mu-1}{r} c_{0,j}^{\mu-1-r} \Big(\sum_{i=1}^{i} c_{i,j}\ell^{i} \Big)^r . \tag{14}$$

The approximate particular solution has the form:

$$\tilde{x}_p(\lambda) = \sum_{j=1}^{m} \tilde{A}_j \int_0^{\lambda} e^{(\lambda-\kappa)\tilde{\omega}_j} f(\kappa)d\kappa . \tag{15}$$

The error of estimation. At the beginning we shall estimate $|A_j-\tilde{A}_j|$ supposing that operators A_j and $\tilde{A}_j$ represent continuous functions for $j = 1,\dots,m$.

Let the inequality

$$\frac{i_0+1}{q} - r + 1 > 1 \tag{16}$$

for $r = 1,\dots,m$; $q \in N$, be satisfied. This enables us to get the error of aproximation of the particular solution better and technically more easier. However, we are able to choose the natural number i_0 in (5) as large as we want, inequality (16) is not restrictive for us.

First, we shall introduce $V_{j,k}$ and $\tilde{V}_{j,k}$ using the paper [2] as:

$$\left|\ell^k \sum_{i=0}^{\infty} c_{i,j}\,\ell^{i/q}\right| < \left|\ell^k \sum_{i=0}^{i_0} c_{i,j}\ell^{i/q}\right| + \tag{17}$$

$$+ \left|\ell^k \sum_{i=i_0+1}^{\infty} c_{i,j}\,\ell^{i/q}\right| < {}_T(\tilde{V}_{j,k} + V_{j,k})\ell;\quad \begin{matrix} k=0 & q=1 \\ k>1 & q>1 \end{matrix}$$

LEMMA 3: *If inequality (16) is satisfied for* $r = 1,\dots,m$ *then we may prove:*

$$\left|\left(\sum_{i=1}^{\infty} c_{i,j}\ell^{i/q}\right)^r - \left(\sum_{i=1}^{i_0} c_{i,j}\ell^{i/q}\right)^r\right| < {}_T V_{j,1-r}\cdot r(\tilde{V}_{j,1} + \tag{18}$$

$$+ V_{j,1})^r \ell^r \equiv E_{j,r}\ell$$

where $\tilde{V}_{j,1}$, $V_{j,1}$ and $V_{j,1-r}$ *are given in (17).*

PROOF:

$$\left|\left(\sum_{i=1}^{\infty} c_{i,j}\ell^{i/q}\right)^r - \left(\sum_{i=1}^{i_0} c_{i,j}\ell^{i/q}\right)^r\right| =$$

$$= \left|\sum_{i=i_0+1}^{\infty} c_{i,j}\ell^{i/q-r+1} \sum_{\gamma=0}^{r-1}\left(\ell\sum_{i=1}^{\infty} c_{i,j}\ell^{i/q}\right)^{\gamma}\left(\ell\sum_{i=1}^{i_0} c_{i,j}\ell^{i/q}\right)\right|$$

$$< {}_T\, V_{j,1-r}\cdot r(\tilde{V}_{j,1} + V_{j,1})^{r-1}\ell^r.$$

LEMMA 4: *If* $W_{j,\mu}$ *and* $\tilde{W}_{j,\mu}$ *are given by (9) and (14), respectively, we have:*

$$(19)\qquad |\ell^{1/q} W_{j,\mu} - \tilde{W}_{j,\mu}| < \sum_{r=1}^{\mu-1} \binom{\mu-1}{r} |c_{0,j}|^{\mu-1-r} E_{j,r}\ell \equiv \varepsilon_{j,\mu}\ell$$

where $E_{j,r}$ *is given in (18).*

PROOF: Indeed:

$$|\ell^{1/q} W_{j,\mu} - \tilde{W}_{j,\mu}| = \Big| \sum_{r=1}^{\mu-1} \binom{\mu-1}{r} \Big(\big(\sum_{i=1}^{\infty} c_{i,j}\ell^{i/q}\big)^r - \big(\sum_{i=1}^{i_0} c_{i,j}\ell^{i/q}\big)^r\Big)\Big| <_T \sum_{r=1}^{\mu-1} \binom{\mu-1}{r} |c_{0,j}|^{\mu-1-r} E_{j,r} :\equiv$$

$$\equiv \varepsilon_{j,\mu}\ell.$$

LEMMA 5: *If* B_j *and* $\tilde{B}_j$ *are given by (10) and (12), respectively, then:*

$$(20)\qquad |B_j - \tilde{B}_j| <_T \sum_{=1}^{m}{}^{*} \sum_{=1}^{n}{}^{*} \frac{\mu\,\alpha_{\mu,\nu}}{\mu_1\alpha_{\mu_1,\nu_1}} \ell^{\nu_1-\nu+p/q(\mu_1-\mu)+1} \varepsilon_{j,\mu}$$

$$\equiv B_j\ell$$

where $\varepsilon_{j,\mu}$ *is given by (19).*

PROOF:

$$|B_j - \tilde{B}_j| = \Big| \sum_{\mu=1}^{m}{}^{*} \sum_{\nu=0}^{n}{}^{*} \frac{\mu\alpha_{\mu,\nu}}{\mu_1\alpha_{\mu_1,\nu_1}} \ell^{\nu_1-\nu+p/q(\mu_1-\mu)} \cdot$$

$$\cdot \Big(\big(\sum_{i=0}^{\infty} c_{i,j}\ell^{i/q}\big)^{\mu-1} - \big(\sum_{i=0}^{i_0} c_{i,j}\ell^{i/q}\big)^{\mu-1}\Big)\Big| < \sum_{\mu=1}^{m}{}^{*} \sum_{\nu=0}^{n}{}^{*} \cdot$$

$$\cdot \Big|\frac{\mu\alpha_{\mu,\nu}}{\mu_1\alpha_{\mu_1,\nu_1}}\Big| \ell^{\nu_1-\nu+p/q(\mu_1-\mu)} \sum_{r=0}^{\mu-1} \binom{\mu-1}{r} |c_{0,j}^{\mu-1-r}| \Big|\Big(\sum_{i=1}^{\infty} c_{i,j}\ell^{i/q}\Big)^r -$$

$$- \left(\sum_{i=0}^{i_0} c_{i,j}\ell^{i/q}\right)^r \Big| \leqslant_T \sum_{\mu=1}^{m}{}^{*} \sum_{\nu=0}^{n}{}^{*} \left|\frac{\mu\alpha_{\mu,\nu}}{\mu_1\alpha_{\mu_1,\nu_1}}\right| \ell^{\nu_1-\nu+p/q(\mu_1-\mu)+1} \cdot$$

$$\cdot\ \varepsilon_{j,\mu} \equiv B_j\ell .$$

Using [2] one can find numbers $\bar{R}_{j,k}(T)$ estimating:

$$(21) \qquad |\ell^{\kappa}(\ell^{1/q} W_{j,\mu_1} + B_j)^k| \leqslant_T \bar{R}_{j,k}(T)\ell, \quad \begin{matrix} \kappa > 1 \text{ if } q > 1 \\ \kappa = 0 \text{ if } q = 1 \end{matrix} .$$

From (13) and (14) follows that

$$(22) \qquad |\ell^{\kappa}(\tilde{W}_{j,\mu_1} + \tilde{B}_j)^k| \leqslant \ell^{\kappa}(|W_{j,\mu_1}| + |B_j|)^k \leqslant_T \bar{R}_{j,k}(T)\ell .$$

There exist numbers $R_j(T)$ and $\tilde{R}_j(T)$ (proposition 1, and papers [1],[2])

$$(23) \qquad \begin{aligned} &\sum_{k=1}^{\infty} \frac{1}{|c_{\mu_1}|^k}\, k\, \bar{R}_{j,k-1}(T) \leqslant_T R_j(T) \\ &\sum_{k=k_0+1}^{\infty} \frac{1}{|c_{\mu_1}|^k}\, \bar{R}_{j,k}(T)\ell \leqslant_T \tilde{R}_j(T) . \end{aligned}$$

PROPOSITION 2: *If* $\nu_1+p/q(\mu_1-1) > 1$ *then the difference between* A_j *and* $\tilde{A}_j$ *given by (11) and (12), respectively, can be estimated as:*

$$(24) \qquad |A_j - \tilde{A}_j| \leqslant_T \frac{(\varepsilon_{j,\mu_1} + B_j)\ell}{|\mu_1\alpha_{\mu_1,\nu_1}c^2_{\mu_1}|} R_j(T) + \tilde{R}_j(T)\ell \equiv A_{\varepsilon,j}\ell$$

where ε_{j,μ_1}, B_j, $R_j(T)$ *and* $\tilde{R}_j(T)$ *are given by (19), (20) and (23) respectively.*

PROOF: The following transformations:

$$|A_j - \tilde{A}_j| = \left|\frac{\ell^{\nu_1+p/q(\mu_1-1)}}{\mu_1\alpha_{\mu_1,\nu_1}c_{\mu}} \sum_{k=0}^{\infty} \frac{(-1)^k}{c^k_{\mu_1}} (\ell^{1/q}W_{j,\mu_1} + B_j)^k - \right.$$

$$- \sum_{k=0}^{k_0} \frac{(-1)^k}{c_{\mu_1}^k} (\tilde{W}_{j,\mu_1} + \tilde{B}_j)^k| < \frac{\ell^{\nu_1+p/q(\mu_1-1)}}{|\mu_1 \alpha_{\mu_1,\nu_1} c_{\mu_1}^2|} \cdot$$

$$\cdot \left(\sum_{k=1}^{\infty} \frac{1}{|c_\mu|^{k-1}} |\ell^{1/q} W_{j,\mu_1} + B_j)^k - (\tilde{W}_{j,\mu_1} + \tilde{B}_j)^k| \right. +$$

$$\left. + \sum_{k=k_0+1}^{\infty} \frac{1}{|c_{\mu_1}|^k} |\tilde{W}_{j,\mu_1} + \tilde{B}_j|^k \right) <$$

$$< \frac{\ell^{\nu_1+p/q(\mu_1-1)}}{|\mu_1 \alpha_{\mu_1,\nu_1} c_{\mu_1}^2|} \left(|\ell^{1/q} W_{j,\mu_1} - \tilde{W}_{j,\mu_1}| + | B_j - \tilde{B}_j| \right) \cdot$$

$$\cdot \sum_{k=1}^{\infty} \frac{1}{c_{\mu_1}^{k-1}} \sum_{d=0}^{k-1} |\ell W_{j,\mu} + B_j|^d \, |\tilde{W}_{j,\mu_1} + \tilde{B}_j|^{k-1-d} +$$

$$+ \sum_{k=k_0+1}^{\infty} \frac{1}{|c_{\mu_1}|^k} |\tilde{W}_{j,\mu} + \tilde{B}_j|^k)$$

using (19), (20), (21), (22) and (23) imply estimations (24).

If operator $f(\kappa)$ represents a continuous function, using [1] and [2], one can find the following approximations:

(25)

$$|f(\kappa)(\exp(\lambda-\kappa)\omega_j - \exp(\lambda-\kappa)\tilde{\omega}_j)| \leq_T \varepsilon x_j(\lambda)$$

$$|f(\kappa) \exp((\lambda-\kappa)\omega_j)| <_T F_{1,j}(\lambda)\ell \qquad 0 < \kappa < \lambda .$$

Finally, approximating $|\tilde{A}_j|$ as:

(26) $$|\tilde{A}_j| <_T R_{1,j}(T)\ell$$

we have:

PROPOSITION 3: *If $f(\kappa)$ is an operator representing a continuous function for $t \geq 0$, then the error of approximation of the particular solution given by (15) instead of (6) is:*

(27) $$|x_p(\lambda) - \tilde{x}_p(\lambda)| \leq_T \sum_{j=1}^{m} (A_{\varepsilon,j}\; F_{1,j}(\lambda)\lambda\ell^2 + \varepsilon x_j(\lambda)\; R_{1,j}\lambda\ell^2)$$

where $A_{\varepsilon,j}, \varepsilon x_j(\lambda)$, $F_{1,j}(\lambda)$ *and* $R_{1,j}$ *are given by (24), (25) and (26).*

PROOF: The difference between $x_p(\lambda)$ and $\tilde{x}_p(\lambda)$ can be transformed as:

$$|x_p(\lambda) - \tilde{x}_p(\lambda)| \leq \Big| \sum_{j=1}^{m} (A_j \int_0^{\lambda} f(\kappa)\exp((\lambda-\kappa)\omega_j)d\kappa -$$

$$- \tilde{A}_j \int_0^{\lambda} f(\kappa)\exp((\lambda-\kappa)\tilde{\omega}_j)d\kappa)\Big| \leq \sum_{j=1}^{m} (\Big|A_j \int_0^{\lambda} f(\kappa)\exp((\lambda-\kappa)\omega_j) -$$

$$- \tilde{A}_j \int_0^{\lambda} f(\kappa)\exp((\lambda-\kappa)\omega_j d\kappa\Big| + \Big|\tilde{A}_j \int_0^{\lambda} f(\kappa)(\exp((\lambda-\kappa)\omega_j) -$$

$$- \exp((\lambda-\kappa)\tilde{\omega}_j))d\kappa\Big| .$$

Using (24), (25) and (26) we have relation (27).

REMARK: If $f(\kappa)$ does not represent a continuous function, but the operator $(x_p(\lambda) - \tilde{x}_p(\lambda))$ represents a continuous function, the error of approximation can be found, using in (25) and (26), the appropriate factors ℓ^k, $k \in N$.

EXAMPLE: Linear partial differential equation with constant coefficients:

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \alpha_{\mu,\nu} \frac{\partial^{\mu+\nu} x(\lambda,t)}{\partial\lambda^{\mu}\, \partial t^{\nu}} = f_1(\lambda,t)$$

with conditions:

$$\frac{\partial^{\mu+\nu} x(\lambda,0)}{\partial\lambda^{\mu}\, \partial t^{\nu}} = \psi_{\mu,\nu}(\lambda) \qquad \mu = 0,\dots,m; \nu = 0,\dots,n$$

$$\frac{\partial^{\mu} x(0,t)}{\partial\lambda^{\mu}} = \varphi_{\mu}(t) \qquad t > 0; \mu = 0,\dots,m-1$$

corresponds in the field M to differential equation (1) with conditions (2), where:

$$f(\lambda) = \{f_1(\lambda,t)\} + \sum_{\kappa=0}^{n-1} s^{n-1-\kappa} \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \alpha_{\mu,n-\kappa+\nu} \frac{\partial^{\mu+\nu} x(\lambda,0)}{\partial\lambda^{\mu}\, \partial t^{\nu}}$$

Now, we shall consider the equation [3]:

(28) $$\frac{\partial^4 x(\lambda,t)}{\partial\lambda^2 \partial t^2} - 2\frac{\partial^3 x(\lambda,t)}{\partial\lambda^2 \partial t} + \frac{\partial^2 x(\lambda,t)}{\partial\lambda} - x(\lambda,t) = 4e^{\lambda}$$

with conditions:

(28´) $$\frac{\partial^2 x(\lambda,0)}{\partial\lambda^2} = e^{\lambda} \qquad \frac{\partial^3 x(\lambda,0)}{\partial\lambda^2 \partial} - 2\frac{\partial^2 x(\lambda,0)}{\partial\lambda^2} = \lambda, \quad \lambda > 0$$

$$x(0,t) = 1 + 2t \qquad \frac{\partial x(0,t)}{\partial\lambda} = 2t^2, \quad t > 0.$$

In the field M this equation corresponds to the equation:

(29) $$(s-1)^2 x''(\lambda) - x(\lambda) = (s-4\ell)e^{\lambda} + \lambda$$

with conditions:

(30) $$x(0) = \ell + 2\ell^2 \qquad x'(0) = 2\ell^2.$$

The characteristic equation of equation (29) is:
$P(\omega) \equiv (s-1)^2\omega^2 - I = 0$ with solutions

$$\omega_1 = \sum_{i=0}^{\infty} \ell^{i+1} \quad \text{and} \quad \omega_2 = -\sum_{i=0}^{\infty} \ell^{i+1}$$

with approximations:

$$\tilde{\omega}_1 = \sum_{i=0}^{i_0} \ell^{i+1} \quad \text{and} \quad \tilde{\omega}_2 = -\sum_{i=0}^{i_0} \ell^{i+1}.$$

However:

$$P'(\omega) = 2(s-1)^2\omega$$

$$P'(\omega_1) = 2(s-2I+\ell)\sum_{i=0}^{\infty}\ell^{i}, \ \nu_1 = 2, \ \mu_1 = 2, \ p/q = -1.$$

From (11) follows:

$$A_1 = \frac{\ell}{2}\sum_{k=0}^{\infty}\Big(\sum_{i=0}^{\infty}\ell^i + (\ell^2 - 2\ell)\sum_{i=0}^{\infty}\ell^i\Big)^k(-1)^k$$

$$A_2 = -\frac{\ell}{2}\sum_{k=0}^{\infty}\Big(\sum_{i=1}^{\infty}\ell^i + (\ell^2 - 2\ell)\sum_{i=0}^{\infty}\ell^i\Big)^k(-1)^k$$

Using (24) we get the estimation:

$$|A_1 - \tilde{A}_1| = |A_2 - \tilde{A}_2| <_T \frac{1}{2}\Big(e^{e^T}|T-1|T\; e^T\Big(\frac{T^{i_0}}{i_0!}|T-1| + e^{k_0T}|T-1|^{k_0+1}\frac{T^{k_0}}{k_0!}\Big)\ell\Big) = A_\varepsilon \ell^2$$

It can be remarked that in this example $f(\lambda) = (s-4\ell)e^\lambda + \lambda$ does not represent a continuous function, but $(x(\lambda) - \tilde{x}(\lambda))$ represents a continuous function, and we are able to find the error of approximation of the particular solution:

$$|x_p(\lambda) - \tilde{x}_p(\lambda)| <_T 2(A_\varepsilon(1 - 2T^2 + \lambda T)e^{\lambda e^T}\ell + R(T)e^{\left(1+T+\frac{T^2}{2}+\dots+\frac{T^{i_0+1}}{(i_0+1)!}\right)}\cdot\frac{T^{i_0}}{i_0!}\,e^T\,e^{e^T}\frac{T^{i_0}}{i_0!}(1-2T^2+\lambda T)\ell).$$

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REZIME

PRIBLIŽNO REŠENJE JEDNE KLASE DIFERENCIJALNIH JEDNAČINA

U radu se posmatra diferencijalna jednačina u polju operatora Mikusinskog:

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \alpha_{\mu,\nu} s^{\nu} x^{(\mu)}(\lambda) = f(\lambda)$$

sa uslovima:

$$x(0) = \varphi_0, \quad x'(0) = \varphi_1, \ldots, x^{(m-1)}(0) = \varphi_{m-1}.$$

Analizira se partikularno rešenje

$$x_p = \sum_{j=1}^{m} A_j \int_{0}^{\lambda} f(\kappa) \exp((\lambda-\kappa)\omega_j) d\kappa$$

konstruiše se približno rešenje i daje ocena greške.