ZBORNIK RADOVA
Prirodno-matematičkog fakulteta
Univerziteta u Novom Sadu
Serija za matematiku, 15,1 (1985)

REVIEW OF RESEARCH
Faculty of Science
University of Novi Sad
Mathematics Series, 15,1 (1985)

ON CONCIRCULAR AND PROJECTIVE CURVATURE TENSORS OF A CERTAIN WEYL-OTSUKI SPACE OF THE SECOND KIND

Nevena Pušić

Prirodno-matematički fakultet, Institut za matematiku, 21000 Novi Sad, dr Ilije Djuričića br.4, Jugoslavija

ABSTRACT

The matter of the paper is the curvature tensor of the covariant part of the general regular connection in a W-O of the second kind; supposing that the tensor is recurrent to Cronecker's delta in the adjoint Riemannian space, we get some interesting relations between concircular and projective curvature tensors in such a W-O and corresponding objects in the adjoint Riemannian space.

INTRODUCTION

Let us consider an n-dimensional Riemannian manifold M with positively defined metrics. Let R^h_{ijk}, R_{ik} and R be the curvature tensor, the Ricci tensor and the scalar curvature of this space.

After applying a projective mapping of M, the coefficients of the Riemannian connection change the following way:

AMS Mathematics Subject Classification (1980):53B15

Key Words and phrases: Regular general connection, curvature tensor, adjoint Riemannian space, projective curvature tensor, concircularity condition, concircular curvature tensor, harmonic vector field, metric connection, one-dimensional Betti number.

$$\bar{\Gamma}^h_{jk} = \{{}_j{}^h{}_k\} + \delta^h_j p_k + \delta^h_k p_j, \tag{0.1}$$

where p_i is a certain gradient vector.

The projective curvature tensor is given by the formula

$$P^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}(\delta^h_k R_{ji} - \delta^h_j R_{ki}). \tag{0.2}$$

We have

PROPOSITION 1. *The projective curvature tensor is invariant under projective mappings.*

The coefficients of conformal connection can be expressed in the form:

$$\bar{\bar{\Gamma}}^h_{jk} = \{{}_j{}^h{}_k\} + \delta^h_j p_k + \delta^h_k p_j - g_{jk} p^h \tag{0.3}$$

where p_i *is a gradient vector.*

The conformal curvature tensor is given by the formula

$$C^h_{ijk} = R^h_{ijk} - \frac{1}{n-1}(R^h_k g_{ij} - R^h_j g_{ik} + R_{ij}\delta^h_k - R_{ik}\delta^h_j) + \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}). \tag{0.4}$$

Then the following holds.

PROPOSITION 2. *The conformal curvature tensor is invariant under conformal mappings.*

If we apply a concircular transformation, the coefficients of connection will change by relation (0.3), but the vector need not to be a gradient obligatorily. It has to satisfy the condition

$$\nabla_k p_h = p_k p_h + \rho g_{hk} \tag{0.5}$$

(ρ is a scalar function), instead. Such a vector is said to be concircular.

The concircular curvature tensor is given by the formula:

$$(0.6)\qquad Z^h_{ijk} = R^h_{ijk} - \frac{R}{n(n-1)}(g_{ij}\,\delta^h_k - q_{ik}\,\delta^h_j).$$

Then, we have

PROPOSITION 3. *The concircular curvature tensor is invariant under concircular mappings.*

Now, we consider a connection whose coefficients are expressed in the form

$$(0.7)\qquad \Gamma^h_{ij} = \{{}_i{}^h{}_j\} + \delta^h_j\, p_i - g_{ij}\, p^h,$$

where p_i is a vector field. This connection is semi-symmetric and metric.

According to [6], we have

PROPOSITION 4.

a) The conformal curvature tensor of the semi-symmetric metric connection is equal to the conformal curvature tensor of the Levi-Civitta connection.

b) If the vector p_i is concircular, the projective curvature tensor of the semi-symmetric metric connection is equal to the projective curvature tensor of the Levi-Civitta connection, and the concircular curvature tensor of the semi-symmetric metric connection is equal to the concircular curvature tensor of the Levi-Civitta connection.

We shall apply these results to the Weyl-Otsuki space of the second kind (denoted by $SW\text{-}O_n$). The covariant differentiation in such a space is given by (for example)

$$T^{i}_{jk,h} = P^{i}_{a} T^{a}_{bc|h} P^{b}_{j} P^{c}_{k}, \tag{0.8}$$

where p^{i}_{j} are coefficients of a linear isomorphism of the space, and $T^{a}_{bc|h}$ is the basic covariant derivative given by two classical affine connections; one of them ($''\Gamma$) works exceptionally on covariant indices and the other one ($'\Gamma$) works on contravariant ones.

SW-O_n are metric spaces and the following conditions hold:

a) $g_{ij,k} = \gamma_k m_{ij}$ (γ_k is a vector field and m_{ij} is a symmetric tensor field).

b) the connection $'\Gamma$ is symmetric

c) $P_{ij} = g_{ia} P^{a}_{j}$ is symmetric.

The inverse of P is denoted by Q. The underlying Riemannian geometry is called the adjoint Riemannian space; its Levi--Civitta covariant differentiation is denoted by $\overset{\circ}{\nabla}$. (For further propositions see [1], [2], [3], [4], [5]).

1. Let us consider the SW-O_n satisfying the condition

$$\overset{\circ}{\nabla}_k P^{i}_{j} = \pi_k \delta^{i}_{j}, \tag{1.1}$$

where π_k is an arbitrary vector field. We shall investigate some features of the connection $''\overset{m}{\Gamma}$ (that is, the covariant part of the metric connection, i.e. $\gamma_k = 0$). According to the basic formulae in [3], the connection $''\overset{m}{\Gamma}$ is an extremely important object in SW-O_n.

We can calculate the coefficients of $''\overset{m}{\Gamma}$:

$$''\overset{m}{\Gamma}{}^{i}_{jk} = \left\{ {i \atop j\ k} \right\} + \tilde{\pi}_j \delta^{i}_{k} - g_{jk} \tilde{\pi}^{i}, \tag{1.2}$$

where $\tilde{\pi}_j$ means the image of π_j, by the isomorphism Q. We can see that $''\overset{m}{\Gamma}$ is a semi-symmetric metric connection.

In [5] there were investigated the facts about the conformal curvature tensor of $''\overset{m}{\Gamma}$ and about its equality to the conformal curvature tensor of the adjoint Riemannian space. We want to make these results complete by applying Proposition 4.b ([6]) to SW-O_n.

We have to chek, at first, when the concircularity condition (0.5) can be satisfied for vector $\tilde{\pi}_i$.

Straight from condition (1.1), we get

$$\overset{\circ}{\nabla}_k Q^i_j = -\pi_k Q^i_a Q^a_j . \tag{1.3}$$

Now

$$\overset{\circ}{\nabla}_k \tilde{\pi}_h = Q^a_h \overset{\circ}{\nabla}_h \pi_a + \pi_a \overset{\circ}{\nabla}_k Q^a_h = Q^a_h \overset{\circ}{\nabla}_k \pi_a - \pi_a \pi_k Q^a_b Q^b_k = \tag{1.4}$$

$$= Q^a_h \overset{\circ}{\nabla}_k \pi_a - \tilde{\tilde{\pi}}_h \pi_k .$$

If we want the vector $\tilde{\pi}_h$ to be concircular, we have to put

$$\overset{\circ}{\nabla}_k \tilde{\pi}_h = Q^a_h \overset{\circ}{\nabla}_k \pi_a - \tilde{\tilde{\pi}}_h \pi_k = \tilde{\pi}_h \tilde{\pi}_k + \rho g_{hk}$$

or, equally

$$Q^a_h \overset{\circ}{\nabla}_k \pi_a = \tilde{\tilde{\pi}}_h \pi_k + \tilde{\pi}_h \tilde{\pi}_k + \rho g_{hk} .$$

Transvecting with P^h_b, we get

$$Q^a_h P^h_b \overset{\circ}{\nabla}_k \pi_a = P^h_b \tilde{\tilde{\pi}}_h \pi_k + P^h_b \tilde{\pi}_h \tilde{\pi}_k + \rho P^h_b g_{hk}$$

and, finally

$$\overset{\circ}{\nabla}_k \pi_b = \tilde{\pi}_b \pi_k + \pi_b \tilde{\pi}_k + \rho P_{bk} . \tag{1.5}$$

So, we have

LEMMA 1. *If vector π_i satisfies condition (1.5), then vector $\tilde{\pi}_i$ is concircular.*

We can easily see that condition (1.5) is trivially satisfied when $\pi_i = 0$. But, if $\pi_i = 0$, the SW-O_n is trivial, it is equal to its adjoint Riemannian space. Now I want to find some nontrivial cases for condition (1.5). There holds

LEMMA 2. *If vector* $\tilde{\pi}_i$ *is concircular, then vector* π_i *is a harmonic vector field.*

PROOF. If vector $\tilde{\pi}_i$ is concircular, then condition (1.5) is sastisfied. We can see immediately

$$\overset{\circ}{\nabla}_{[k} \ \pi_{b]} = 0,$$

because the tensor P_{bk} is symmetric. After that,

$$\overset{\circ}{\nabla}_k \ \pi^k = \tilde{\pi}^k \ \pi_k + \pi^k \ \tilde{\pi}_k + \rho P_k^k = 2\tilde{\pi}^k \ \pi_k + \rho \mathrm{tr} P.$$

We can always choose the scalar function ρ in such a way, that $\overset{\circ}{\nabla}_k \ \pi^k = 0$. Hence the vector is harmonic.

According to a famous result of Salomon Bochner ([7]), we get

COROLLARY 1. *If the one-dimensional Betti number of the manifold* M *is different from zero, the covariant metric connection on SW-O_n satisfying condition (1.1) can be a concirculary semi-symmetric metric connection.*

According to Proposition 4.b we have the next theorem

THEOREM 1. *In SW-O_n, if conditions (1.1) and (1.5) are satisfied*

$$(1.6) \qquad \overset{m}{\overset{\leftharpoonup\leftharpoonup}{R}}{}^h_{ijk} - \frac{\overset{\leftharpoonup\leftharpoonup}{\overset{m}{R}}}{n(n-1)} (g_{ij} \ \delta^h_k - g_{ik} \ \delta^h_j) =$$

$$= R^h_{ijk} - \frac{R}{n(n-1)} (g_{ij} \ \delta^h_k - g_{ik} \ \delta^h_j) .$$

After lowering the upper index h, we get the covariant components of curvature tensors. Then, we have

$$(1.7)\qquad {}''\overset{m}{R}_{hijk} = R_{hijk} - \frac{R - {}''\overset{m}{R}}{n(n-1)}\,(g_{ij}\, g_{hk} - g_{ik}\, g_{hj}).$$

COROLLARY 2. *In* $SW\text{-}O_n$, *if conditions (1.1) and (1.5) are satisfied, the covariant components of curvature tensor* $''\overset{m}{R}$ *are skew-symmetric in the first two indices and invariant under changing places of the first and the second pair of indices.*

COROLLARY 3. *If the adjoint Riemannian space is concircularly flat, then the components of curvature tensor* $''\overset{m}{R}$, *under conditions (1.1) and (1.5) can be expressed in the following way*

$${}''\overset{m}{R}{}^h_{ijk} - \frac{{}''\overset{m}{R}}{n(n-1)}\,(g_{ij}\,\delta^h_k - g_{ij}\,\delta^h_j)$$

in any point of the manifold.

Except for this, we have

THEOREM 2. *In* $SW\text{-}O_n$, *if conditions (1.1) and (1.5) are satisfied,*

$$(1.8)\qquad {}''\overset{m}{R}{}^h_{ijk} - \frac{1}{n-1}\,(\delta^h_k\, {}''\overset{m}{R}_{ji} - \delta^h_j\, {}''\overset{m}{R}_{ik}) =$$

$$= R^h_{ijk} - \frac{1}{n-1}\,(\delta^h_k\, R_{ji} - \delta^h_j\, R_{ik})$$

Theorem 1 and its corollaries are necessary conditions for the statement of Theorem 2. Then tensor $''\overset{m}{R}_{ij}$ is symmetric (it is obtained by contraction $''\overset{m}{R}{}^h_{ijk}$).

COROLLARY 4. If the adjoint Riemannian space is projectively flat, then the components of curvature tensor $''\overset{m}{R}$ (under conditions (1.1) and (1.5)) can be expressed in the following way

$$''\overset{m}{R}{}^h_{ijk} = \frac{1}{n-1}\left(\delta^h_k\, ''\overset{m}{R}_{ji} - \delta^h_j\, ''\overset{m}{R}_{ik}\right).$$

One could give another proof of Theorem 2 directly from Theorem 1.

In any case of $\overset{\circ}{\nabla}_k P^i_{\cdot j} = \pi_k \overset{m}{P}{}^i_{\cdot j}$, the symmetric connection $'\overset{m}{\Gamma}$ of SW-O_n can neither be projective nor conformal, unless $P^i_{\cdot j} = \rho\delta^i_j$ ([4]).

REFERENCES

[1] M. Prvanović, *Weyl-Otsuki spaces of the second and third kind, Zbornik radova Prirodno-matematičkog fakulteta u Novom Sadu, 11 (1981), 219 - 226.*

[2] M. Prvanović, *On a special connection in an Otsuki space, Tensor, N. S. 37 (1982), 237 - 243.*

[3] N. Pušić, *Sveżnjevi i opšte koneksije na mnogostrukostima, magistarski rad, PMF Novi Sad, 1982.*

[4] N. Pušić, *Weyl-Otsuki spaces of the second kind with a special tensor P, Zbornik radova Prirodno-matematičkog fakulteta u Novom Sadu, 12 (1982), 379 - 386.*

[5] N. Pušić, *A new kind of a metric Otsuki space, Zbornik radova Prirodno-matematičkog fakulteta u Novom Sadu, 14, 1 (1984), 63 - 71.*

[6] W. Słósarska, *On some invariants of Riemannian manifolds admitting a concircularly semi-symmetric metric connection, Demonstratio mathematica, XVII (1984), 251 - 257.*

[7] K. Yano and S. Bochner, *Curvature and Betti numbers, Princeton, New York, 1953.*

Received by the editors June 19, 1985.

REZIME

O KONCIRKULARNOM I PROJEKTIVNOM TENZORU KRIVINE NEKIH WEYL-OTSUKIJEVIH PROSTORA DRUGE VRSTE

U radu se daju neke specijalne osobine tenzora krivine kovarijantnog dela opšte regularne koneksije Vejl-Ocukijevog prostora druge vrste kod koga je tenzor P rekurentan Kronekerovom simbolu. Ova koneksija je semisimetrična metrička, te se odatle mogu dobiti osobine tenzora projektivne i koncirkularne krivine i posebno, uslov koncirkularnosti.