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SOME FIXED POINT THEOREMS IN PROBABILISTIC METRIC SPACES

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ABSTRACT

In [7] a new class of contraction type mappings on probabilistic metric spaces is introduced and a fixed point theorem for such mappings is proved. V. Radu generalized in [14] the fixed point theorem from [7] to (S,F,t), which is a complete Menger space with T-norm t such that $\sup_{a<1} t(a,a) = 1$. In this paper we shall generalize fixed point theorems from [7], [14] and [2].

1. INTRODUCTION

The notion of probabilistic metric space is introduced in [8] and some fixed point theorems in such spaces are proved in [2], [3], [5], [7], [10], [11], [12], [13], [14], [17], [18].

Let S be a nonempty set, Δ the family of distribution functions, $F : S \times S \to \Delta$ and t a T-norm [15].

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DEFINITION. *The triplet* (S,F,t) *is a Menger space if and only if the following conditions are satisfied, where* $F_{p,q} = F(p,q)$, *for every* $(p,q) \in S \times S$:

(i) $F_{p,q}(0) = 0$, *for every* $(p,q) \in S \times S$.

(ii) $F_{p,q}(x) = 1$, *for all* $x > 0$ *if and only if* $p = q$.

(iii) $F_{p,q} = F_{q,p}$, *for every* $(p,q) \in S \times S$.

(iv) *For every* $(p,q,r) \in S \times S \times S$ *and* $x,y \in R^+$:

$$F_{p,q}(x+y) \geq t(F_{p,r}(x), F_{r,q}(y)).$$

The (ε,λ)-topology is introduced by the family of neighbourhoods $V = \{V_u(\varepsilon,\lambda) | (u,\varepsilon,\lambda) \in S \times R^+ \times (0,1)\}$. This topology is metrizable if $\sup_{a<1} t(a,a) = 1$, which is proved in [16].

The notion of a probabilistic contraction type mapping is introduced by V. Sehgal in [17] and for such class of mappings in [18] a fixed point theorem is proved, where T-norm t is min. In [6] it is proved that this result holds if t has equicontinuous iterations at x = 1 (see also [11]).

T. Hicks introduced in [7] the following contraction condition, where $k \in (0,1)$:

(1) For every $r > 0$, $F_{p,q}(r) > 1-r$ implies $F_{fp,fq}(kr) > 1-kr$ where $f:S \to S$, $(p,q) \in S \times S$ and (S,F,t) is a Menger space (in [7] the case when t = min is investigated).

In [14] V. Radu proved the following result.

THEOREM A. *Let* (S,F,t) *be a complete Menger space such that* $\sup_{a<1} t(a,a) = 1$. *Then every mapping* $f:S \to S$ *which satisfies condition (1) has a unique fixed point* $x \in S$ *and* $x = \lim_{n\to\infty} p_n$, *where* $p_0 \in S$ *and* $p_{n+1} = fp_n$, *for every* $n \in \mathbb{N} \cup \{0\}$.

In [14] it is proved that condition (1) implies that for every $(p,q) \in S \times S$:

$$d(f^n p, f^n q) \leq 2k^n d(p,q), \text{ for every } n \in N$$

if $t(a,b) \geq \max\{a+b-1,0\}$, $(a,b) \in [0,1] \times [0,1]$ and d is defined by:

$$(2) \qquad d(p,q) = \inf_{t>0}\{t+1-F_{p,q}(t)\}, \quad (p,q) \in S \times S.$$

The metric d, which is defined by (2), induces the (ε,λ)-uniformity [14].

In part 2. of this paper we shall prove a generalization of Theorem A. For Theorem 2 of this paper we shall need the following theorem of P.R. Meyers [9].

THEOREM B. *Let (X,d) be a complete metric space, $f: X \to X$ a continuous mapping and the following conditions are satisfied:*

(a) f has a unique fixed point x^*.

(b) For each $x \in X$ the sequence $\{f^n x\}_{n \in N}$ converges to x^*.

(c) There exists an open neighbourhood U of x^* with the property that for any given open set V including x^* there is an $n_0 \in N$ such that $n \geq n_0$ implies $f^n(U) \subset V$.

Then for each $k \in (0,1)$ there exists a metric d^*, topologically equivalent to the metric d, so that:

$$d^*(fp,fq) \leq kd^*(p,q), \quad (p,q) \in S \times S.$$

2. FIXED POINT THEOREMS

The following theorem is a generalization of Theorem A.

THEOREM 1. *Let* (S,F,t) *be a complete Menger space such that* $\sup_{a<1} t(a,a) = 1$, $f:S \to S$ *and for any* $x \in S$ *there exists* $n(x) \in \mathbb{N}$ *such that for any* $v \in O_f(x;0,\infty) = \{f^n x, n \in \mathbb{N} \cup \{0\}\}$:

$$(3) \qquad r > 0,\ F_{x,v}(r) > 1-r \Rightarrow F_{f^{n(x)}x,\,f^{n(x)}v}(g(r)) > 1-g(r)$$

where $g: [0,\infty) \to [0,\infty)$ *is such that* $\lim_{n\to\infty} g^n(r) = 0 (r > 0)$ *and* $g(u)<u, u > 0$.

If f *is continuous, then there exists* $x^* \in S$ *such that* $fx^* = x^*$. *If* (3) *holds for every* $(x,v) \in S \times S$, *then there exists one and only one fixed point* $x^* \in S$ *of* f *and* $x^* = \lim_{n\to\infty} f^n x_o$, *for arbitrary* $x_o \in S$.

PROOF. Let $x_o \in S$ and $x_n = f^{n(x_{n-1})} x_{n-1}$, $n \in \mathbb{N}$. We shall prove that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence which means that for every $r > 0$ and $s \in (0,1)$ there exists $n(r,s) \in \mathbb{N}$ so that:

$$F_{x_{m+p},x_m}(r) > 1-s, \quad \text{for every } p \in \mathbb{N}$$

and every $m \geq n(r,s)$.

For every $r > 0$ we have:

$$F_{f^{n(x_{m+p-1})}\cdots f^{n(x_m)}x_o,\,x_o}(1+r) > 1 - (1+r).$$

Since

$$v = f^{n(x_{m+p-1})} \cdots f^{n(x_m)} x_o \in O_f(x_o;0,\infty)$$

from (3) we have that:

$$F_{f^{n(x_0)} f^{n(x_{m+p-1})} \dots f^{n(x_n)} x_0, f^{n(x_0)} x_0}(g(1+r)) > 1-g(1+r).$$

Similarly, for $x = x_1$ and $v = f^{n(x_{m+p-1})} \dots f^{n(x_n)} x_1 \in O(x_1; 0,\infty)$ we have:

$$F_{f^{n(x_{m+p-1})} \dots f^{n(x_m)} f^{n(x_1)} x_1, f^{n(x_1)} x_1}(g^2(1+r)) > 1-g^2(1+r).$$

It is easy to see that for every $p \in N$ and every $m \in N$

$$F_{f^{n(x_{m+p-1})} \dots f^{n(x_m)} x_{m-1}, x_{m-1}}(g^{m-1}(1+r)) > 1-g^{m-1}(1+r).$$

Since $\lim_{n\to\infty} g^n(r) = 0$, for every $r > 0$ it follows that there exists $n_o(r,s) \in N$ so that:

$$g^n(1+r) < \min\{ \quad s\}, \text{ for every } n \geq n_o(r,s).$$

Hence we have that for every $m > n_o(r,s)$ and $p \in N$:

$$F_{f^{n(x_{m+p-1})} \dots f^{n(x_m)} x_{m-1}, x_{m-1}}(r) \geq$$

$$\geq F_{f^{n(x_{m+p-1})} \dots f^{n(x_m)} x_{m-1}, x_{m-1}}(g^{m-1}(1+r)) > 1-g^{m-1}(1+r) > 1-s$$

which means that $\{x_n\}_{n\in N}$ is a Cauchy sequence. Since S is complete there exists x^* such that $\lim_{n\to\infty} x_n = x^*$.

Let us prove that $\lim_{n\to\infty} fx_n = x^*$. First, we shall prove that for every $r > 0$ and every $s \in (0,1)$ there exists $n_1(r,s) \in N$ so that:

$$F_{fx_m, x_m}(r) > 1-s, \text{ for every } m \geq n_1(r,s).$$

From $F_{fx_o,x_o}(1+r) > 1-(1+r)$ it follows that:

$$F_{fx_m,x_m}(g^m(1+r)) > 1-g^m(1+r), \text{ for every } m \in \mathbf{N}.$$

If $n_o(r,s) \in \mathbf{N}$ is such that $g^m(1+r) < \min\{r,s\}$ for every $m \geq n_o(r,s)$ then:

(4) $$F_{fx_m,x_m}(r) > 1-s, \text{ for every } m \geq n_o(r,s)=n_1(r,s).$$

Since for every $u > 0$:

$$F_{fx^*,x^*}(u) \geq t(t(F_{fx^*,fx_m}(u/3),F_{fx_m,x_m}(u/3)),F_{x_m,x^*}(u/3))$$

and $\sup_{a<1} t(a,a) = 1$, it follows from (4) and the continuity of f that $fx^* = x^*$.

Suppose that (3) holds for every $(x,v) \in S \times S$. Let us prove that $f^{n(x^*)}x^* = x^*$. Let $r > 0$. Then from

$$F_{f^{n(x^*)}x_o,x_o}(1+r) > 1-(1+r)$$

it follows that for every $m \in \mathbf{N}$ we have that

$$F_{f^{n(x^*)}x_m,x_m}(g^m(1+r)) > 1-g^m(1+r).$$

Let $s \in (0,1)$ and $n_o(r,s) \in \mathbf{N}$ so that for every $n \geq n_o(r,s)$, $g^n(1+r) < \min\{r,s\}$. Then for every $n \geq n_o(r,s)$

$$F_{f^{n(x^*)}x_n,x_n}(r) > 1-s$$

and so $\lim_{n\to\infty} x_n = x^*$ implies that $\lim_{n\to\infty} f^{n(x^*)}x_n = x^*$.

The topology of the space S is Hausdorff. Hence, in order to prove that $x^* = f^{n(x^*)}x^*$ we shall prove that $\lim_{n\to\infty} f^{n(x^*)}x_n = f^{n(x^*)}x^*$.

Let $r > 0$, $s \in (0,1)$ and suppose that $0 < u < \min\{r,s\}$. From $g(u) < u$ it follows that $g(u) < \min\{r,s\}$. From $\lim_{n\to\infty} x_n = x^*$ we obtain that there exists $n(u) \in \mathbf{N}$ so that $F_{x_n,x^*}(u) > 1-u$, for every $n \geq n(u)$ and so from (3) we have:

$$F_{f^{n(x^*)}x_n,\ f^{n(x^*)}x^*}(g(u)) > 1-g(u), \text{ for every } n \geq n(u).$$

Hence:

$$F_{f^{n(x^*)}x_n,\ f^{n(x^*)}x^*}(r) \geq F_{f^{n(x^*)}x_n,\ f^{n(x^*)}x^*}(g(u)) >$$

$$> 1-g(u) > 1-s, \text{ for every } n \geq n(u)$$

which means that $\lim_{n\to\infty} f^{n(x^*)}x_n = f^{n(x^*)}x^*$, and so $f^{n(x^*)}x^* = x^*$. Let us prove the uniqueness of the fixed point of the mapping $f^{n(x^*)}$. Suppose that $v \in S$ so that $f^{n(x^*)}v = v$. From $F_{x^*,v}(1+r) > 1-(1+r)$ for every $r > 0$ it follows that:

$$F_{f^{n(x^*)}x^*,\ f^{n(x^*)}v}(g(1+r)) > 1-g(1+r)$$

and so:

(5) $$F_{x^*,v}(g^n(1+r)) > 1-g^n(1+r), \text{ for every } n \in \mathbf{N}.$$

From (5) and $\lim_{n\to\infty} g^n(1+r) = 0$ we obtain that $x^* = v$. Since $x^* = f^{n(x^*)}x^*$ we have that:

$$fx^* = ff^{n(x^*)}x^* = f^{n(x^*)}fx^*$$

and so $fx^* = x^*$. Let us prove that $x^* = \lim_{n\to\infty} f^n x_0$. For every $r > 0$ and $k \in \{0,1,\ldots, n(x^*)-1\}$:

$$F_{f^k x_0, x^*}(1+r) > 1-(1+r)$$

which implies that:

$$F_{f^{k+n(x^*)}x_0, f^{n(x^*)}x^*}(g(1+r)) > 1-g(1+r)$$

and so:

$$F_{f^{k+n(x^*)}x_0, x^*}(g(1+r)) > 1-g(1+r).$$

It is obvious that for every $m \in \mathbf{N}$:

$$F_{f^{mn(x^*)+k}x_0, x^*}(g^m(1+r)) > 1-g^m(1+r)$$

and so for every $n \geq n(x^*)$:

(6) $$F_{f^n x_0, x^*}(g^{[n/n(x^*)]}(1+r)) > 1-g^{[n/n(x^*)]}(1+r).$$

Relation (6) implies that $\lim_{n\to\infty} f^n x_0 = x^*$.

REMARK. From the proof of Theorem 1 it is obvious that we can suppose that $\inf_r g(r) = 0$ instead of $g(r) < r (r > 0)$. If $n(x) = 1$, for every $x \in S$ from Theorem 1 it follows Theorem A.

THEOREM 2. *Let* (S,F,t) *be a complete Menger space such that* $\sup_{a<1} t(a,a) = 1$ *and* $f:S \to S$ *a continuous mapping so that (3) is satisfied for every* $(x,v) \in S \times S$. *Then for each* $k \in (0,1)$ *there exists a metric* d^**, topologically equivalent to a metric* d *which induces the* (ε,λ)*-uniformity, such that:*

(7) $$d^*(fp,fq) \leq kd^*(p,q), \text{ for every } (p,q) \in S \times S.$$

PROOF. (a) and (b) from Theorem B are satisfied and let us prove (c), where $U = S$ and $V = V_{x^*}(r,s)$ $(r > 0,\ s \in (0,1))$. We shall prove that there exists $n(r,s) \in \mathbf{N}$ so that for every $n \geq n(r,s)$, $f^n(U) \subset V$. Let $p \in U$ and $n(r,s) \in \mathbf{N}$ so that $g^n(1+r) <$ $< \min\{r,s\}$, for every $n \geq n(r,s)$. For every $n \geq n(x^*)$ we have that:

$$F_{f^n p, x^*}(g^{[n/n(x^*)]}(1+r)) > 1-g^{[n/n(x^*)]}(1+r).$$

If $[n/n(x^*)] > n(r,s)$ then $F_{f^n p, x^*}(r) > 1-s$ and so $f^n p \in V$, for every $n \in \mathbb{N}$ such that $[n/n(x^*)] > n(r,s)$.

Let $(S, F, \min)$ be a Menger space and for every $(x,y) \in S \times S$:

$$d(x,y) = \begin{cases} \sup\{t \in (0,1),\ F_{x,y}(t) \le 1-t\} \\ 0,\ F_{x,y}(t) > 1-t \text{ for every } t \in (0,\infty). \end{cases}$$

In [7] ([2]) it is proved that d is a metric on S which is compatible with the (ε,λ)-topology. It is obvious that $d(x,y) < t$ if and only if $F_{x,y}(t) > 1-t$. Using the metric d Shihsen Chang proved in [2] some fixed point theorems in a Menger space $(S, F, \min)$. The following theorem is a generalization of Theorem 3.3 from [2].

THEOREM 3. *Let* (S, F, t) *be a complete Menger space such that* $\sup_{a<1} t(a,a) = 1$, $f: S \to S$ *a continuous mapping and for any* $x \in S$ *there exists* $p(x) \in \mathbb{N}$ *such that for any* $u,v \in O_f(x;0,\infty)$ *and any* $r > d(u,v)$:

$$F_{f^{p(x)}u,\ f^{p(x)}v}(g(r)) > 1-g(r)$$

where $g: [0,\infty) \to [0,\infty)$ *is nondecreasing right continuous and* $g(r) < r$, *for every* $r > 0$. *Then there exists a fixed point of* f *and for any* $x_0 \in S$ *the sequence* $\{f^n x_0\}_{n\in\mathbb{N}}$ *converges to some fixed point of* f.

PROOF. Let us prove that $\lim_{n\to\infty} g^n(r) = 0$, for $r > 0$. From $g(r) < r$ it follows that $g^n(r) \le g^{n-1}(r) \le \ldots < r$ and so there exists $r^* = \lim_{n\to\infty} g^n(r)$. Since g is right continuous it follows that $\lim_{n\to\infty} g^{n+1}(r) = g(\lim_{n\to\infty} g^n(r)) = g(r^*)$ and so $r^* = 0$. Let us prove that for every $x_0 \in S$, the sequence $\{f^n x_0\}_{n\in \mathbf{N}}$ converges to a fixed point of the mapping f. First, we shall prove that the sequence $\{f^{kn(x_0)}x_0\}_{k\in \mathbf{N}}$ is convergent. Since for every $r>0$,

$$F_{f^{pn(x_0)}x_0,x_0}(1+r) > 1-(1+r),$$

for every $p \in \mathbf{N}$ we obtain that for every $k \in \mathbf{N}$:

$$F_{f^{(k+p)n(x_0)}x_0,\ f^{kn(x_0)}x_0}(g^k(1+r)) > 1-g^k(1+r).$$

Hence if $s \in (0,1)$ and $n(r,s) \in \mathbf{N}$ is such that:

$$g^k(1+r) < \min\{r,s\},\ \text{for } k \geq n(r,s)$$

then:

$$F_{f^{(k+p)n(x_0)}x_0,\ f^{kn(x_0)}x_0}(r) > 1-s,\ \text{for every } k\geq n(r,s)$$

and so $\{f^{n(x_0)k}x_0\}_{k\in \mathbf{N}}$ is a Cauchy sequence. Let $x^* = \lim_{k\to\infty} f^{n(x_0)k}x_0$. Let $p \in \{1,2,\ldots, n(x_0)-1\}$ and $r > 0$. Since

$$F_{f^p x_0,x_0}(1+r) > 1-(1+r)$$

we obtain that for every $k \in \mathbf{N}$

$$F_{f^{kn(x_0)+p}x_0,\ f^{kn(x_0)}x_0}(g^k(1+r)) > 1-g^k(1+r)$$

and so we have $\lim_{k\to\infty} f^{kn(x_0)+p}x_0 = x^*$. This implies that $x^* = \lim_{n\to\infty} f^n x_0$ and $fx^* = x^*$.

REMARK. If t = min from Theorem 3 it follows Theorem 3.3 from [2]. Similarly, the following theorem can be proved.

THEOREM 4. *Let* (S, F, t) *be a complete Menger space such that* $\sup_{a<1} t(a,a) = 1$ *and* $f: S \to S$ *a continuous mapping so that the following condition is satisfied: There exists* $p \in N$ *so that for every* $x \in S$ *and every* $k \in N$

$$F_{f^p x,\ f^{p+k} x}(g(r)) > 1 - g(r),\ \textit{for every}\quad r > d(x, f^k x).$$

where the function g *is as in Theorem 3.*
Then the conclusion of Theorem 3 remains true.

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REZIME

NEKE TEOREME O NEPOKRETNOJ TAČKI U VEROVATNOSNIM METRIČKIM PROSTORIMA

U radu [7] definisana je nova klasa kontraktivnih preslikavanja u verovatnostnim metričkim prostorima i dokazana je teorema o nepokretnoj tački za ova preslikavanja.

V. Radu je uopštio u radu [14] teoremu o nepokretnoj tački iz rada [7] na Mengerove prostore (S,F,t) gde je

$$\sup_{a<1} t(a,a) = 1.$$

U ovom radu uopšteni su rezultati iz [7], [14], i [2].