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THREE COUNTERFEIT COINS

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ABSTRACT

We consider the problem of ascertaining the minimum number of weighings which suffice to determine all counterfeit (heavier) coins in a set of n coins of the same appearance, given a balance scale and the information that there are exactly three heavier coins present. An optimal procedure is constructed for an infinite set of n's and a suboptimal procedure for another infinite set of n's. We also consider a slightly modified problem, i.e., the case when we are given a certain number (not greater than n) of additional coins for which we know that they are all good (not counterfeit). For that case, and arbitrary n, we determine an upper bound for the maximum number of steps (weighings) of an optimal procedure which differs by just two from the information-theoretical lower bound. The proof is given by an effective construction of a procedure.

1. INTRODUCTION

Let $X=\{c_1,c_2,\dots,c_n\}$ be a set of n coins indistinguishable except that exactly m of them are slightly heavier than the rest . We suppose that all heavier coins are of equal weight, and so are all light (good) coins. If λ is the weight of a light coin, then the weight of a heavy (counterfeit) coin is less than $\frac{m+1}{m}\lambda$, so that the larger of two numerically unequal subsets of X is always the heavier.

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Given a balance scale, we want to find an optimal weighing procedure, i.e. a procedure which minimizes the maximum number of steps (weighings) which are required to identify all heavier coins. It is clear that no information is gained by balancing two numerically une ual sets. We also suppose that the scale reveals which, if either, of two subsets of X is heavier but not by how much.

Step (A,B) will mean the balancing of A against B, where A and B are disjoint subsets of X of the same cardinality. The possible outcomes are:

(a) $A = B$ (the sets balance),

(b) $A \neq B$ (the sets do not balance).

We use the notation $A < B$, $A > B$, where $<$ and $>$ between two sets mean "is lighter than" and "is heavier than" respectively.

If $A \subseteq X$, $h(A) = t$ will mean that A contains exactly t heavier coins, while $|A|$ denotes the cardinality of the set A.

By $P_n^m(\ell)$ we denote any procedure which enables us to identify all heavier coins, if there are exactly m of them in the set of n coins, ℓ being the maximal number of steps to be required. $P_n^m(\leq \ell)$ will mean a procedure for which the maximal number of steps to be required is not greater than ℓ.

A procedure $P_n^m(\ell)$ is said to be optimal if no one procedure $P_n^m(r)$ exists for some $r < \ell$. We write $\mu_m(n) = \ell$ if there is an optimal procedure $P_n^m(\ell)$. A procedure $P_n^m(\ell)$ is said to be suboptimal if $\mu_m(n) = \ell-1$. It follows by information-theoretical reasonings that $\mu_m(n) \geq \lceil \log_3 \binom{n}{m} \rceil$, where $\lceil x \rceil$ denotes the least integer $\geq x$. Remark that for any procedure $P_n^m(\ell)$, we have a dual procedure $P_n^{n-m}(\ell)$, because by identifying m heavier coins, we also identify n-m lighter coins.

It is well known that $\mu_1(n) = \lceil \log_3 n \rceil$. In [6] it is proved that

$$\lceil \log_3 \binom{n}{2} \rceil \leq \mu_2(n) \leq 1 + \lceil \log_3 \binom{n}{2} \rceil$$

and a corresponding procedure is constructed such that the lower bound is reached for an infinite set of n's. In [7] it is proved that for $n = 3^k$ $(k = 2, 3, \ldots)$

$$\lceil \log_3 \binom{n}{3} \rceil \leq \mu_3(n) \leq 1 + \lceil \log_3 \binom{n}{3} \rceil$$

but the optimality of the constructed procedure is proved only for $k = 2$.

In this paper, some new results concerning the problem of three counterfeit coins are obtained.

2. THE RESULTS

THEOREM 1. *If* $n = 3^k + 3^{k-1}$, *where* k *is any positive integer, then*

$$\mu_3(n) = \lceil \log_3 \binom{n}{3} \rceil .$$

P r o o f. It is easy to check that $3^{3k-1} < \binom{3^k+3^{k-1}}{3} <$ $< 3^k$, i.e., $\left\lceil \log_3 \binom{3^k+3^{k-1}}{3} \right\rceil = 3k$, for $k \geq 3$. Now, the statement will be proved by the inductive construction of a procedure $P^3_{3^k+3^{k-1}}(\leq 3k)$, for $k \geq 1$.

For $k = 1$, the construction of a procedure $P^3_4(2)$ is trivial, and it is an optimal procedure.

Suppose that a procedure $P^3_{3^{k-1}+3^{k-2}}(\leq 3k-3)$ is constructed. Then, a procedure $P^3_{3^k+3^{k-1}}(\leq 3k)$ can be constructed as follows.

Let $X, |X| = 3^k+3^{k-1}$, be partitioned into four subsets, i.e., let $X = A \cup B \cup C \cup D$, where $|A| = |B| = |C| = |D| = 3^{k-1}$ and A, B, C and D are disjoint. It is clear that $h(A) + h(B) +$ $+ h(C) + h(D) = 3$.

Step 1. (A,B).

Step 2. (C,D).

It suffices to consider two cases ((a) and (b) below); because of symmetry, any other possible case is quite analogous to one of these two.

(a) $A=B, C<D$.

Step 3. (A,C).

(aa) If $A<C$, then $h(C)=1$ and $h(D)=2$. We continue by successive application of two procedures, $P^1_{3^{k-1}}(k-1)$ and $P^2_{3^{k-1}}(2k-2)$, to the sets C and D respectively. The construction of the procedure $P^2_{3^k}(2k)$, for arbitrary k, is given in [6].

(ab) If $A=C$, then $h(D)=3$. Let $A'\subset A$, such that $|A'|=3^{k-2}$. The, $|D\cup A'|=3^{k-1}+3^{k-2}$ and $h(D\cup A')=3$. We apply a procedure $P^3_{3^{k-1}+3^{k-2}}(\leqslant 3k-3)$ to the set $D\cup A'$. This procedure can be constructed by the induction hypothesis.

(ac) If $A>C$, then $h(A)=h(B)=h(D)=1$. We continue by successive application of a procedure $P^1_{3^{k-1}}(k-1)$ three times, to the sets A, B and D independently.

In each case, all heavier coins will be found after at most 3k steps.

(b) $A<B$, $C<D$.

Step 3. (B,D).

The outcome $B=D$ is not possible.

(ba) If $B<D$, then $h(B)=1$ and $h(D)=2$.

(bb) If $B>D$, then $h(B)=2$ and $h(D)=1$.

In both cases, we continue quite similarly as in the case (aa).

A procedure $P^3_{3^k+3^{k-1}}(\leq 3k)$ is constructed. The theorem is proved.

REMARK 1. It is clear that the constructed procedure, for $k \geq 3$, is in fact an optimal procedure $P^3_{3^k+3^{k-1}}(3k)$. On the other hand, we are not sure that, for $k=2$, the constructed procedure $P^3_{12}(6)$ is optimal, because $\lceil \log_3 \binom{12}{3} \rceil = 5$. It is an open question whether a procedure $P^3_{12}(5)$ exists.

THEOREM 2. *If* $n=2.3^k$, *for any integer* $k \geq 2$, *then*

$$\lceil \log_3 \binom{n}{3} \rceil \leq \mu_3(n) \leq 1 + \lceil \log_3 \binom{n}{3} \rceil .$$

P r o o f. It is easy to check that $3^k < \binom{2\cdot 3^k}{3} < 3^{k+1}$, i.e. $\lceil \log_3 \binom{2\cdot 3^k}{3} \rceil = 3k+1$, for $k \geq 2$. So, the statement will be proved by construction of a procedure $P^3_{2\cdot 3^k}(3k+2)$, for $k \geq 2$.

Let X, $|X| = 2\cdot 3^k$, be partitioned into six subsets, i.e. $X = A \cup B \cup C \cup D \cup E \cup F$, where $|A| = |B| = |C| = |D| = |E| = |F| = 3^{k-1}$. It is clear that $h(A) + h(B) + h(C) + h(D) + h(E) + h(F) = 3$.

Step 1. (A,B).
Step 2. (C,D).
Step 3. (E,F).

Because of symmetry, it suffices to consider three cases ((a), (b) and (c) below); any other possible case is quite analog to one of these three.

(a) $A < B$, $C < D$, $E < F$. We conclude that $h(B) = h(D) = h(E) = 1$, and continue by successive application of a procedure $P^1_{3^{k-1}}(k-1)$ three times, to the sets B, D and E respectively. So, all heavier coins will be found after $3k$ steps.

(b) $A < B$, $C < D$, $E = F$. We conclude that $h(B) \geq 1$, $h(D) \geq 1$ and $h(B \cup D) = 3$.

Step 4. (B,D).

There are two possibilities:

(ba) $B < D$. Now, we know that $h(B) = 1$ and $h(D) = 2$, and we continue by successive application of two procedures, $P^1_{3^{k-1}}(k-1)$ and $P^2_{3^{k-1}}(2k-2)$, to the sets B and D respectively. So, all heavier coins will be found after 3k+1 steps.

(bb) $B > D$. Now, we know that $h(B) = 2$ and $h(D) = 1$, and continue quite similarly as in (ba).

(c) $A < B$, $C = D$, $E = F$.

Step 4. (C,E)

There are three possibilities:

(ca) $C < E$. We conclude that $h(B) = h(E) = h(F) = 1$, and we continue quite similarly as in (a). All heavier coins will be found after 3k+1 steps.

(cb) $C > E$. This case is quite analog to (ca).

(cc) $C = E$. Now, we know that $h(A \cup B) = 3$.

Step 5. (A,C)

There are two possibilities:

(cca) $A = C$. We conclude that $h(B) = 3$. Now, we apply a procedure $P^3_{3^{k-1}+3^{k-2}}(3k-3)$ to the set $B \cup A'$, where $A' \subset A$ and $|A'| = 3^{k-2}$. Such a procedure is constructed in the proof of Theorem 1. So, all heavier coins will be found after 3k+2 steps.

(ccb) $A > C$. Now, we know that $h(A) = 1$ and $h(B) = 2$, and we continue quite similarly as in (ba). All heavier coins will be found after 3k+2 steps.

A procedure $P^3_{2\cdot 3^k}(3k+2)$ is constructed. The theorem is proved.

REMARK 2. For $n = 18$, i.e. for $k = 2$, the procedures $P^3_{3^{k-1}+3^{k-2}}(3k-3)$ and $P^2_{3^{k-1}}(2k-2)$ used in (cca) and (ccb) can be replaced by procedures $P^3_4(2)$ and $P^2_3(1)$ respectively. So, in this case we obtain an optimal procedure $P^3_{18}(7)$. The case $n = 6$ $(k = 3)$ is not considered in the theorem, but it is easy to construct an optimal procedure $P^3_6(3)$. For all other numbers $n = 2\cdot 3^k$ $(k \geq 3)$, we only know that the constructed procedure is either optimal or suboptimal.

3. A MODIFICATION OF THE COUNTERFEIT COINS PROBLEM

Suppose that in addition to the given set $X = \{c_1, c_2, \ldots, c_n\}$, containing exactly m counterfeit coins, we have at our disposal a sufficiently large number of coins for which we know that they are all good (not counterfeit). Step (A,B) will mean the balancing of A against B, where A and B are two sets of the same cardinality and each of them may contain some of additional good coins.

In such a modified problem we use the notation $P'^m_n(\ell)$ and $\mu'_m(n)$ instead of $P^m_n(\ell)$ and $\mu_m(n)$ respectively. It is clear that $\mu'_m(n) \leq \mu_m(n)$.

THEOREM 3. *Let* $n \geq 3$. *Then*

$$\lceil \log_3 \binom{n}{3} \rceil \leq \mu'_3(n) \leq 2 + \lceil \log_3 \binom{n}{3} \rceil .$$

P r o o f. If $n = 3^k + 3^{k-1}$ or $n = 2\cdot 3^k$, $k \geq 1$, the statement follows from theorems 1 and 2.

If $2\cdot 3^{k-1} < n < 3^k + 3^{k-1}$, we add $3^k + 3^{k-1} - n$ good coins to the set X, producing a set X′ containing $3^k + 3^{k-1}$ coins. Now, for $k \geq 2$, we construct a procedure $P'^3_{3^k+3^{k-1}}(\leq 3k)$ as in Theorem 1.

The statement follows since $\lceil \log_3 \binom{n}{3} \rceil \geq 3k-2$. For $k=1$, a procedure $P'^{3}_{4}(2)$ can be constructed.

If $3^k + 3^{k-1} < n < 2\cdot 3^k$, we add $2\cdot 3^k - n$ good coins to the set X, producing a set X′ containing $2\cdot 3^k$ coins. Now, for $k > 3$, we construct a procedure $P'^{3}_{2\cdot 3^k}(3k+2)$ as in Theorem 2. The statement follows since $\lceil \log_3 \binom{n}{3} \rceil \geq 3k$. For $k=1$ and $k=2$, we construct procedures $P'^{3}_{6}(3)$ and $P'^{3}_{18}(7)$ respectively.

The theorem is proved.

REMARK 3. It is clear that we need at most n-2 additional good coins if $2\cdot 3^{k-1} < n < 3^k + 3^{k-1}$, $k > 2$, and at most $\frac{n-3}{2}$ additional good coins if $3^k + 3^{k-1} < n < 2\cdot 3^k$, $k > 1$.

REMARK 4. In the proofs of Theorem 1 (see (ab)) and Theorem 2 (see (cca)) we use some coins already identified as good coins in order to enlarge the set containing three heavier coins. So, instead of a set containing 3^{k-1} coins, we use an enlarged set containing $3^{k-1}+3^{k-2}$ coins. We do so because the construction of a procedure $P^{3}_{3^k}(3k)$ (given in [7]) seems to be more complicated than the construction of a procedure $P^{3}_{3^k+3^{k-1}}(3k)$. This fact justifies the following question:

Problem. Are $\mu_3(n)$ and $\mu'_3(n)$ increasing functions of n?

More generaly, the question of monotonicity of the functions $\mu_m(n)$ and $\mu'_m(n)$ is open. From results given in [6] it follows that for $m=2$, the answer is affirmative.

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REZIME

TRI NEISPRAVNA NOVČIĆA

U radu je konstruisano ternarno stablo koje predstavlja algoritam za identifikaciju tri neispravna (teža) elementa iz skupa od n elemenata. Algoritam je optimalan za jedan beskonačan skup vrednosti parametara n dok je za preostale vrednosti od n taj algoritam skoro optimalan.