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FUZZY CONGRUENCE RELATIONS AND GROUPOIDS

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ABSTRACT

Fuzzy congruence relations on algebras are defined in [3], and discussed in [2],[4] and [5]. Fuzzy groupoids are considered in [1],[5], and [2].

In this paper, we define a weak fuzzy congruence relation on a groupoid, and we prove that this relation uniquely determines a fuzzy groupoid on the same algebra. Starting with a weak fuzzy congruence relation $\bar{\rho}$ on a groupoid (S,.) and using the decomposition of a fuzzy set defined in [2], we get two collections of fuzzy sets (on S, and on S^2, respectively). We prove that the first collection consists of the fuzzy groupoids on (S,.), and that in the second are the fuzzy congruence relations on these groupoids.

All fuzzy sets here are L-valued, where L is a complete lattice.

1. Let $S \neq \emptyset$, and let $L = (L, \wedge, \vee, 0, 1)$ be a complete lattice. $\bar{\rho} : S^2 \to L$ is a <u>weak fuzzy equivalence relation on S</u>, iff the following conditions are satisfied:

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(2) $\bar{\rho}(x,y) = \bar{\rho}(y,x)$, for all $x,y \in S$;

(3) $\bar{\rho}(x,y) \geq \bigvee_{z \in S} (\bar{\rho}(x,z) \wedge \bar{\rho}(z,y))$, for all $x,y \in S$.

REMARK. An obvious consequence of (2) and (3) is (1) $\bar{\rho}(x,x) \geq \bar{\rho}(x,y)$, for all $x,y \in S$, and if $\bar{\rho}(x,x) = 1$ for every $x \in S$, then $\bar{\rho}$ is a fuzzy equivalence relation on S ([3]).

Let $\bar{A} : S \to L$ be a fuzzy set on S, and let $\bar{\rho} \subseteq \bar{A}^2$ (that is $\bar{\rho}(x,y) \leq \bar{A}(x) \wedge \bar{A}(y)$, for all $x,y \in S$, [6]). $\bar{\rho}$ is a <u>fuzzy equivalence relation on $\bar{A}$</u>, iff it satisfies (2), (3) and

$$\bar{\rho}(x,x) = \bar{A}(x), \text{for every } x \in S .$$

Let $(S,\cdot)$ be a groupoid. A <u>fuzzy groupoid</u> $(\bar{A},\cdot)$ on $(S,\cdot)$ is a mapping $\bar{A} : S \to L$, satisfying

(4) $\bar{A}(x \cdot y) \geq \bar{A}(x) \wedge \bar{A}(y)$, for all $x,y \in S$ ([1]).

Let $(S,\cdot)$ be a groupoid ($(\bar{A},\cdot)$ a fuzzy groupoid on $(S,\cdot)$), and let $\bar{\rho}$ be a weak fuzzy equivalence relation on S (a fuzzy equivalence relation on $\bar{A}$) satisfying the substitution property:

(5) $\bar{\rho}(x \cdot u, y \cdot v) \geq \bar{\rho}(x,y) \wedge \bar{\rho}(u,v)$, for all $x,y,u,v \in S$.

Then $\bar{\rho}$ is a <u>weak fuzzy congruence relation on $(S,\cdot)$</u> (<u>a fuzzy congruence relation on $(\bar{A},\cdot)$</u>).

PROPOSITION 2.1. *If $\bar{\rho}$ is a weak fuzzy equivalence relation on $\bar{A} : S \to L$, theh $\bar{\rho}$ is a weak fuzzy equivalence relation on S.*

P r o o f. From

$\bar{\rho}(x,x) = \bar{A}(x)$, it follows that

$\bar{\rho}(x,x) \leq \bar{A}(x) \wedge \bar{A}(y) \leq \bar{A}(x) = \bar{\rho}(x,x)$. □

PROPOSITION 2.2. *If $\bar{\rho}$ is a weak fuzzy equivalence relation on S, and if $\bar{A} : S \to L$, such that*

$$\bar{A}(x) = \bar{\rho}(x,x), \quad \textit{for every} \quad x \in S,$$

then $\bar{\rho}$ is a fuzzy equivalence relation on $\bar{A}$.

P r o o f. From

$$\bar{\rho}(x,y) \leq \bar{\rho}(x,x) = \bar{A}(x), \quad \text{and}$$

$$\bar{\rho}(x,y) \leq \bar{\rho}(y,y) = \bar{A}(y), \quad \text{it follows that}$$

$$\bar{\rho}(x,y) \leq \bar{A}(x) \wedge \bar{A}(y). \quad \square$$

Now we shall prove that a weak fuzzy congruence relation on a groupoid $(S,\cdot)$ induces on $(S,\cdot)$ a fuzzy groupoid $(\bar{A},\cdot)$ (which is also called a fuzzy subgroupoid of $(S,\cdot)$).

THEOREM 2.3. *Let $\bar{\rho}$ be a weak fuzzy congruence relation on $(S,\cdot)$. Then, a mapping $\bar{A}: S \to L$, defined with*

$$\bar{A}(x) = \bar{\rho}(x,x) \quad , \text{for every } x \in S,$$

is a fuzzy groupoid on $(S,\cdot)$, and $\bar{\rho}$ is a fuzzy congruence relation on $(\bar{A},\cdot)$.

P r o o f. Since $\bar{A}(x) = \bar{\rho}(x,x)$, and $\bar{A}(y) = \bar{\rho}(y,y)$, the following is satisfied:

$$\bar{A}(x\cdot y) = \bar{\rho}(x\cdot y, x\cdot y) \geq \bar{\rho}(x,x) \wedge \bar{\rho}(y,y) = \bar{A}(x) \wedge \bar{A}(y).$$

Thus, $\bar{A}(x\cdot y) \geq \bar{A}(x) \wedge \bar{A}(y)$, proving that $(\bar{A},\cdot)$ is a fuzzy groupoid on $(S,\cdot)$.

By the definition, $\bar{\rho}$ is a fuzzy congruence relation on $(\bar{A},\cdot)$. $\square$

REMARK. If $L = \{0,1\}$, then the last theorem gives that (a nonempty) symmetric and transitive relation ρ on a groupoid $(S,\cdot)$, satisfying the substitution property, determines a subgroupoid $(A,\cdot)$ of $(S,\cdot)$.

3. In [2] it was proved that a fuzzy set $\bar{A}: S \to L$ uniquely determines a family $\{\bar{A}p \mid p \in L\}$ of fuzzy sets on S, and vice versa. The theorems of decomposition and synthesis of $\bar{A}$ by means of that family were also given. From there we have:

$$\bar{A}p(x) \overset{def}{=} \begin{cases} \bar{A}(x), & \text{if } A(x) \geq p \\ 0, & \text{otherwise}. \end{cases} \quad (x \in S)$$

We shall now consider the fuzzy congruence relation $\bar{\rho}$ on a fuzzy groupoid $(\bar{A},\cdot)$ on $(S,\cdot)$, and also the corresponding families of fuzzy relations and sets.

PROPOSITION 3.1. *Let $(\bar{A},\cdot)$ be a fuzzy groupoid on $(S,\cdot)$, and let $\bar{\rho}$ be a fuzzy congruence relation on $(\bar{A},\cdot)$. Let $\{\bar{A}p \mid p \in L\}$, and $\{\bar{\rho}p \mid p \in L\}$ be such that*

$$\bar{A} = \bigcup_{p>0} \bar{A}p, \text{ and } \bar{\rho} = \bigcup_{p>0} \bar{\rho}p$$

([2]), then for every $p \in L$:

a) $(\bar{A}p,\cdot)$ is a fuzzy groupoid of $(S,\cdot)$, and

b) $\bar{\rho}p$ is a fuzzy congruence relation on $(\bar{A}p,\cdot)$.

P r o o f. a) If $\bar{A}p(x) = 0$, or $\bar{A}p(y) = 0$, $x,y \in S$, then clearly $\bar{A}p(x\cdot y) \geq \bar{A}p(x) \wedge \bar{A}p(y)$.

Suppose now that $\bar{A}p(x) \neq 0$, and $\bar{A}p(y) \neq 0$. Then

$\bar{A}p(x) = \bar{A}(x) \geq p$, and $\bar{A}p(y) = \bar{A}(y) \geq p$. Hence

$\bar{A}(x\cdot y) \geq \bar{A}(x) \wedge \bar{A}(y) \geq p$, and thus

$\bar{A}p(x\cdot y) \geq \bar{A}p(x) \wedge \bar{A}p(y)$, proving that $(\bar{A}_p,\cdot)$ is fuzzy groupoid on $(S,\cdot)$.

b) $\bar{\rho}p$ is a fuzzy relation on $\bar{A}p$:
Indeed if $(x,y) \in S^2$, then $\bar{\rho}(x,y) \geq p$, or $\bar{\rho}(x,y) \not\geq p$. In the first case,

$$p \leq \bar{\rho}(x,y) = \bar{\rho}p(x,y) \leq \bar{A}(x) \wedge \bar{A}(y) = \bar{A}p(x) \wedge \bar{A}p(y).$$

If $\bar{\rho}(x,y) \not\geq p$, then $\bar{\rho}p(x,y) = 0$, and the inequality is satisfied again.

Now we shall prove that $\bar{\rho}p$ is a fuzzy equivalence relation on $\bar{A}p$.

$\bar{\rho}p$ is obviously reflexive and symmetric. To prove that it is transitive, consider the supremum

$$\bigvee_{z\in S} (\bar{\rho}p(x,z) \wedge \bar{\rho}p(z,y)) \qquad (i)$$

If it is equal to zero, then $\bar{\rho}p$ is transitive. If not, take all infima $\bar{\rho}p(x,z) \wedge \bar{\rho}p(z,y)$ in which both values are not equal to zero (i.e. they are not less than p). Then

$$p < \bar{\rho}p(x,z) \wedge \bar{\rho}p(z,y) = \bar{\rho}(x,z) \wedge \bar{\rho}(z,y) < \bar{\rho}(x,y) = \bar{\rho}p(x,y).$$

The same inequality is satisfied by the supremum (i), proving that $\bar{\rho}p$ is transitive.

$\bar{\rho}p$ satisfies the substitution property (5): Consider $\bar{\rho}p(x,y)$, and $\bar{\rho}p(u,v)$, $x,y,u,v \in S$, $p \in L$. If at least one of these values is 0, condition (5) is directly satisfied.

Suppose now that $\bar{\rho}p(x,y) \neq 0$, and $\bar{\rho}p(u,v) \neq 0$. Then

$$\bar{\rho}(x,y) = \bar{\rho}p(x,y) > p, \text{ and } \bar{\rho}(u,v) = \bar{\rho}p(u,v) > p ,$$

and thus

$$\bar{\rho}(x\cdot u, y\cdot v) > \bar{\rho}(x,y) \wedge \bar{\rho}(u,v) > p.$$

Thereby,

$$\bar{\rho}p(x\cdot u, y\cdot v) = \bar{\rho}(x\cdot u, y\cdot v),$$

and $\bar{\rho}p$ satisfies (5). □

Applying the synthesis theorems of fuzzy sets and relations formulated in [2], on the fuzzy groupoids and the corresponding (weak) fuzzy congruence relations, we get the following two propositions.

PROPOSITION 3.2. *Let* $\{\bar{A}p;\ p \in L\}$ *be a family of fuzzy sets on* S *satisfying:*

a) $\bar{A}p(x) \in \{0\} \cup [p)$, *for every* $x \in S$; ($[p)$ *is a filter (generated by* $p \in L$*)*

b) If $s,t \in L$, *and* $s < t$, *then:*

b1) $\bar{A}t(x) \neq 0$ *implies* $\bar{A}s(x) = \bar{A}t(x)$;

b2) If $\bar{A}s(x) = t$, *then* $\bar{A}t(x) = t$.

Also, let for every $p \in L$, $(\bar{A}_p,.)$ *be a fuzzy subgroupoid of a groupoid* $(S,.)$.

Then $(\bar{A},.)$, *where* $\bar{A}$ *is defined as in Proposition 3.1, is a fuzzy groupoid on* $(S,.)$.

PROPOSITION 3.3. *Let* $\{\bar{\rho}p;\ p \in L\}$ *be a family of fuzzy relations on* S, *satisfying conditions (a) and (b), and for every* $p \in L$, *let* $\bar{\rho}p$ *be a fuzzy congruence relation on a fuzzy subgroupoid* $(\bar{A}p,.)$ *of a groupoid* $(S,.)$.

Then

$$\bar{\rho} = \bigcup_{p>0} \bar{\rho}p$$

is a fuzzy congruence relation on a fuzzy subgroupoid $(\bar{A},.)$ *of a groupoid* $(S,.)$, *where*

$$\bar{A} = \bigcup_{p>0} \bar{A}p \ .$$

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REZIME

RASPLINUTE KONGRUENCIJE I GRUPOIDI

U radu je definisana slaba rasplinuta relacija kongruencije na proizvoljnom grupoidu i dokazano je da ta relacija jednoznačno odredjuje rasplinuti podgrupoid datog grupoida. Pokazano je da se postupkom dekompozicije rasplinutog grupoida i odgovarajuće rasplinute kongruencije na njemu dobijaju familije podgrupoida i rasplinutih kongruencija na njima.