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ASYMPTOTICS OF SOLUTIONS OF A CLASS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT

The asymptotics of solutions tending to zero for $x\to\infty$ of the equation $y''= f(g(x))\phi(y)$ is studied with $f(g(x))$ of unrestricted growth.

1. INTRODUCTION

In the present paper asymptotic properties of solutions tending to zero for $x\to\infty$ of the equation

$$(1.1)\qquad y'' = f(x)\phi(y)$$

will be considered; f and ϕ are positive and continuous on (a,∞). In contrast to the results of [1], [2], [3], the growth of $f(x)$ is here not restricted at all.

Asymptotic estimates for solutions in question are also obtained by Taliaferro [4], under different hypotheses (even more general concerning the growth of $\phi(y)$).

We shall give here the precise asymptotic behaviour of solutions by extending the method outlined in [5] in a special case of $f(x)$.

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To formulate the results we shall recall the definitions and some properties of slowly and regularly varying functions of Karamata [6], and of one of their generalizations introduced by Avakumović [7], and independently by Bari and Stečkin [8].

DEFINITION 1.1. *A function* $g(x)$ *is said to be almost increasing (decreasing) if there exists an* $A > 1$ *such that* $x_1 < x_2$ $(x_1 > x_2)$ *implies* $g(x_1) < A\,g(x_2)$.

DEFINITION 1.2. *A positive continuous function* $g(x)$ *defined on* (a,∞) *is said to be O-regularly varying (O-r.v.f.) at infinity (or: to belong to the class* $R_\infty(\gamma,\Gamma)$*) if there exist real numbers* γ, Γ *such that* $x^\gamma g(x)$ *almost increases and* $x^{-\Gamma} g(x)$ *almost decreases for large* x.

DEFINITION 1.3. *A positive continuous function* $L(x)$ *defined on* (a,∞) *is said to be slowly varying (s.v.f.) at infinity if for all* $\omega > 0$

$$\lim_{x\to\infty} \frac{L(\omega x)}{L(x)} = 1 .$$

A function $L_1(x)$ *is s.v.f. at zero if* $L_1(1/x)$ *is s.v.f. at infinity.*

DEFINITION 1.4. *The function*

$$g(x) = x^\sigma L(x)$$

is said to be o-regularly varying (o-r.v.f.) at infinity. The real number σ *is called the index of regularity.*

For the sake of completeness, we shall present here the needed properties of functions in question, the proof of which can be found in [7] and [9].

PROPOSITION 1.1. *If* $\sigma > -1$, *then*

$$\int_a^x t^{\sigma} L(t)dt \sim (\sigma+1)^{-1} x^{\sigma+1} L(x), \quad x \to \infty ;$$

if $\sigma = -1$, $\ell(x) = \int_a^x t^{-1} L(t)dt$ *is slowly varying and* $L(x)/\ell(x) \to 0$, $x \to \infty$.

PROPOSITION 1.2. *If for some* $\eta > 0$

$$\int_a^{\infty} t^{\eta} |f(t)| dt < \infty$$, *then*

$$\int_a^{\infty} f(t) L(xt)dt \sim L(x) \int_a^{\infty} f(t)dt, \quad x \to \infty .$$

PROPOSITION 1.3. *If* $\sigma > -1$, *then*

$$\int_0^y t^{\sigma} L(t)dt \sim (\sigma+1)^{-1} y^{\sigma+1} L(y), \quad y \to 0 .$$

PROPOSITION 1.4. *If* $\sigma < -1$, *then*

$$\int_y^a t^{\sigma} L(t)dt \sim (-\sigma-1)^{-1} y^{\sigma+1} L(y), \quad y \to 0 .$$

PROPOSITION 1.5. *If* $g(x)$ *is o-regularly varying of index* $\sigma > 0$, *then its inverse (if it exists) is also such, and of index* $1/\sigma$.

PROPOSITION 1.6. *If both* $g_1(x)$ *and* $g_2(x)$ *are o-regularly varying of index* σ_1, σ_2, *respectively, then the function* $g_1\{g_2(x)\}$ *is also o-regularly varying and of index* $\sigma_1\sigma_2$. *In that,* $g_2(x)$ *is assumed to tend either to infinity or to zero, depending on which of the two points* $g_1(x)$ *is defined in.*

PROPOSITION 1.7. *If* $g(x) \sim x^{\sigma} L(x)$, $x \to \infty$, *then* $g(x) = x^{\sigma} L^*(x)$, *where* $L(x) \sim L^*(x)$, $x \to \infty$.

PROPOSITION 1.8. *If* $xy'(x) / y(x) \to 0$, $x \to \infty$, *then* $y(x)$ *is slowly varying at infinity.*

PROPOSITION 1.9. *If* $L(x)$ *is slowly varying at infinity, then, for each* $\varepsilon > 0$, $x^{\varepsilon}L(x)$ *almost increases and* $x^{-\varepsilon}L(x)$ *almost decreases for* $x \geq x_0$.

2. ASYMPTOTIC BEHAVIOUR OF SOLUTIONS

THEOREM 2.1. *Let* $f(x)$ *be o-r.v.f. at* ∞ *of index* $\sigma > 0$ *and let* $\phi(y)$ *be o-r.v.f. at zero of index* $\lambda > 1$. *Further, let* $g(x)$ *be positive, continuously differentiable and tending to infinity with* x *and such that* $g'(x)/g(x) = \ell(g(x))$ *, where* $\ell(t)$ *is s.v.f. at* ∞. *Then, for each positive solution* $y(x)$ *tending to zero for* $x \to \infty$ *of the equation*

(2.1) $$y'' = f(g(x))\phi(x) ,$$

there holds for $x \to \infty$

(2.2) $$\frac{\phi(y(x))}{y(x)} \sim \{\frac{\sigma}{\lambda-1}\frac{g'(x)}{g(x)}\}^2\{f(g(x))\}^{-1} .$$

Notice that the formula (2.2) can be, due to Definition 1.4, rewritten as

(2.2′) $$y^{\lambda-1}(x)L_1(x) \sim (\frac{\sigma}{\lambda-1})^2\{\frac{g^{\sigma+2}(x)}{g'^2(x)}L(g(x))\}^{-1}, \quad x \to \infty.$$

The answer to the question how to obtain the behaviour of $y(x)$ from (2.2) now depends on the form of $\phi(y)$ or, due to (2.2′), of $L_1(y)$. In fact, the considered solutions of (2.1) are of the form

(2.3) $$y(x) = \{g(x)\}^{\sigma/(1-\lambda)}L_2\{g(x)\} ,$$

where $L_2(t)$ is s.v.f. implicitly defined by

(2.4) $$L_2^{\lambda-1}\{g(x)\}L_1\{g^{\sigma/(1-\lambda)}(x)L_2(g(x))\} \sim$$

$$\sim (\frac{\sigma}{\lambda-1})^2\ell^2(g(x))L^{-1}(g(x)) , \quad x \to \infty .$$

We shall illustrate Theorem 2.1 by the following example.

$$y'' = \Gamma(x) y^{\lambda} \ell n^{\eta} \frac{1}{y}, \quad \lambda > 1.$$

Since $\Gamma(x) \sim e^{x(\ell nx-1)} \sqrt{2\pi x}$, $x \to \infty$, we take $g(x) = e^{x(\ell nx-1)}$, so that the function $\sqrt{2\pi x}$ is s.v.f. in $g(x)$ according to Proposition 1.8. Hence, due to Proposition 1.7, $\Gamma(x) = g(x)L(g(x))$, where $L(g(x)) \sim \sqrt{2\pi x}$, $x \to \infty$. Thus by (2.2´), (2.3) and (2.4)

$$y(x) \sim \{(\lambda-1)^{2-\eta} \frac{x^{\eta}(\ell nx-1)^{\eta}}{(\ell nx)^2} \Gamma(x)\}^{1/(1-\lambda)}, \quad x \to \infty.$$

P r o o f of Theorem 2.1. First notice that $g(\omega x) / g(x) \to \infty$, $x \to \infty$ for all $\omega > 1$. For,

$$\int_a^x \frac{g'(t)dt}{g(t)\ell(g(t))} = x - a$$

and, due to Proposition 1.1, the left-hand side integral is s.v.f. $\bar{\ell}(g(x))$, such that $\bar{\ell}(g(x))\ell(g(x)) \to \infty$, $x \to \infty$, i.e.

$$\frac{xg'(x)}{g(x)} \to \infty, \quad x \to \infty. \tag{2.5}$$

Hence, $g'(x)/g(x) > m/x$ for $x \geq x_o$ and so, by integrating over $(x, \omega x)$, $g(\omega x)/g(x) > \omega^m$ for $x \geq x_o$.

On the other hand, the inverse function of $g(x)$, call it $h(x)$, tends to infinity and, moreover, from (2.5), $xh'(x)/h(x) \to 0$, $x \to \infty$. Hence, by Proposition 1.8, $h(x)$ is s.v.f. at infinity.

Let

$$J(x) = \int_a^x dt \int_a^t f(g(u))du,$$

$$z(x) = J(x) \int_o^{y(x)} \frac{dt}{t} \int_o^t \frac{\phi(u)}{u^2} du. \tag{2.6}$$

By putting $g(u) = v$, and by applying Proposition 1.1 twice, we obtain

$$J(x) \sim \frac{1}{\sigma^2} \{\frac{g(x)}{g'(x)}\}^2 f\{g(x)\}, \quad x \to \infty. \tag{2.7}$$

Furthermore,

$$(2.8)\qquad J'(x) = \int_a^x f(g(t))dt \sim \frac{1}{\sigma}\,\frac{g(x)}{g'(x)}\, f\{g(x)\},\ x \to \infty ,$$

$$(2.9)\qquad J''(x) = f\{g(x)\}.$$

Also for $y \to 0$, from Proposition 1.3, there follows

$$(2.10)\qquad \int_0^y \frac{\phi(t)}{t^2}\,dt \sim \frac{1}{\lambda-1}\,\frac{\phi(y)}{y} ,\quad \int_0^y \frac{dt}{t}\int_0^t \frac{\phi(u)}{u^2}\,dt \sim \frac{1}{(\lambda-1)^2}\,\frac{\phi(y)}{y} ,$$

hence, by using (2.6), (2.7) and (2.10),

$$(2.11)\qquad z(x) \sim \left\{\frac{1}{\sigma(\lambda-1)}\,\frac{g(x)}{g'(x)}\right\}^2 f(g(x))\frac{\phi(y(x))}{y(x)} ,\ x \to \infty .$$

Because of (2.1), the function $z(x)$ satisfies the equation

$$(2.12)\qquad \frac{z''}{z}\,\frac{J}{J'} = 2\frac{z'}{z} + \frac{J''}{J} - 2\,\frac{J'}{J} + z\,\frac{J''}{J'}v_1 + \left(\frac{z'}{z} - \frac{J'}{J}\right)^2 \frac{J}{J'}v_2 ,$$

where

$$(2.13)\qquad v_1 = \frac{\phi(y)}{y}\int_0^y \frac{\phi(t)}{t^2}\,dt\left\{\int_0^y \frac{dt}{t}\int_0^t \frac{\phi(u)}{u^2}\,du\right\}^{-2} ,$$

$$(2.14)\qquad v_2 = \left\{\frac{\phi(y)}{y} - \int_0^y \frac{\phi(t)}{t^2}\,dt\right\}\int_0^y \frac{dt}{t}\int_0^t \frac{\phi(u)}{u^2}\,du\left\{\int_0^y \frac{\phi(t)}{t^2}\,dt\right\}^{-2} .$$

From (2.10), (2.13) and (2.14), there follows

$$(2.15)\qquad v_1(y) = (\lambda-1)^3 + o(1),\quad v_2(y) = \frac{\lambda-2}{\lambda-1} + o(1),\quad y \to 0.$$

Then, because of (2.7), (2.8), (2.9) and (2.15), equation (2.12) is reduced to

$$\{\lambda-1+o(1)\}z'' - \{\lambda-2+o(1)\}\,\frac{z'^2}{z} - \{2\sigma + o(1)\}\,\frac{g'(x)}{g(x)}\,z' =$$

$$(2.16)\qquad = \sigma^2\,\frac{g'^2(x)}{g^2(x)}\,z\{z((\lambda-1)^4 + o(1)) - 1 + o(1)\} ,$$

where $o(1)$ denotes various functions tending to zero for $x \to \infty$.

We shall prove that $z(x)$ tends to a positive constant when $x \to \infty$: The asymptotic equality (2.16) shows that the maxima of $z(x)$ (when $z'(x) = 0$, $z''(x) < 0$) cannot be at a finite distance and above the line $y = (\lambda-1)^{-4}$, and similarly the minima if $z(x)$ cannot be at a finite distance and below that line. Hence, $z(x)$ is either ultimately monotone or its extremes tend to $(\lambda-1)^{-4}$, $x \to \infty$.

We still have to show that in the former case $z(x)$ tends to a positive constant. Clearly, it is sufficient to show that there exist numbers C_1, C_2 $(C_2 > C_1 > 0)$ such that

(2.17) $$0 < C_1 \leq z(x) \leq C_2, \quad x \geq x_o .$$

To that end, first notice that Definitions 1.1 and 1.4 together imply that the functions $f(t)$ and $\phi(y)$ satisfy the following conditions:

For $t \to \infty$

(2.18) $t^{-p} f(t)$ almost increases for some $p > 0$,

(2.19) $t^{-q} f(t)$ almost decreases for some $q \geq p > 0$.

For $y \to 0$

(2.20) $y^{-r}\phi(y)$ almost decreases for some $r > 1$,

(2.21) $y^{-s}\phi(y)$ almost increases for some $s \geq r > 1$.

Next assume that $z(x)$ increases for $x \geq x_o$. By integrating (2.1) over (x,∞), one obtains for any $\theta > 0$

(2.22) $$-y'(x) = \int_x^\infty f(g(t))\phi(y(t)) \frac{z^{1+\theta}(t)}{z^{1+\theta}(t)}\, dt$$

or, using (2.11),

(2.23) $$-y'(x) \geq m \int_x^\infty \frac{\{y(t)z(t)\}^{1+\theta}}{\{\phi(y(t))f(g(t))\}^{\theta}} \left\{\frac{g'(t)}{g(t)}\right\}^{2(1+\theta)} dt .$$

Here, and further on in the text, all (positive) minorizing constants will be denoted by the same letter m, and the majorizing ones by M, unless the exact value is needed. Now, by virtue of (2.19) and (2.20), from (2.23) there follows the existence of a $\theta>0$, such that $1+\theta(1-r)<0$; whence for $x\geqslant x_o$

$$(2.24)\qquad -y'(x)\geqslant m\left\{\frac{g^q(x)}{\phi(y(x))f(g(x))}\right\}^{\theta}\{y(x)z(x)\}^{1+\theta}$$

$$\times\int_x^{\infty} g^{-q\theta}(t)\left\{\frac{g'(t)}{g(t)}\right\}^{2(1+\theta)}dt\ .$$

By putting $g(t)=u$ and applying Proposition 1.9 , the integral in (2.24) is minorized by $m\,g^{-q\theta}(x)\left\{\frac{g'(x)}{g(x)}\right\}^{1+2\theta}$. Also, $\{y(x)z(x)\}^{1+\theta}$ is minorized by using (2.11). Thus (2.24) is reduced to

$$(2.25)\qquad -\frac{y'(x)}{\phi(y(x))}\geqslant m\,\frac{g(x)}{g'(x)}\,f(g(x)),\quad x\geqslant x_o\ .$$

By integrating both sides of (2.25) over (a,x), and then, applying to the left-hand side integral condition (2.20), and to the right-hand side one Proposition 1.9. and (2.19), one gets

$$\frac{y(x)}{\phi(y(x))}\geqslant m\left\{\frac{g(x)}{g'(x)}\right\}^{2}f(g(x)),\quad x\geqslant x_o\ .$$

From the above inequality, by virtue of (2.11), one concludes that there exists a $C_2>0$ such that $z(x)\leqslant C_2$. The left-hand side inequality in (2.17) is obvious, since z(x) increases.

Next let us assume that z(x) decreases for $x\geqslant x_o$. Then the right-hand side inequality in (2.17) is obvious. Also, $z(x)\to c_1(\geqslant 0)$, $x\to\infty$. We shall show that $c_1>0$. To do that we proceed as in the previous case by majorizing $-y'(x)$ in (2.22), using (2.18), (2.21) and by choosing, this time, $\theta>0$ such that $1+\theta(1-s)>0$. This gives for $x\geqslant x_o$

$$(2.26)\qquad -y'(x)\leqslant M\,y(x)\,z(x)\,\frac{g'(x)}{g(x)}\ .$$

On the other hand, we differentiate both sides of (2.6),and apply (2.20), (2.21) and (2.26) to obtain

(2.27) $$\frac{J'}{J} - \frac{z'}{z} < M\, z(x)\, \frac{g'(x)}{g(x)}\,, \quad x \geq x_o \,.$$

It follows that z(x) cannot tend to zero for $x \to \infty$. Otherwise, one would have that the right-hand side of (2.12) would be negative for $x \geq x_o$, due to (2.7), (2.8), (2.9), (2.15) and (2.27). But then one would have $z''(x) < 0$ and z(x), being positive, cannot tend to zero. Whence, $z(x) \to c_1 > 0, x \to \infty$.

Consequently, z(x) always tends to a positive constant for $x \to \infty$, which we have to determine. To that end let us write (2.11) as

(2.28) $$\frac{\phi(y(x))}{y(x)} \sim \{\sigma(\lambda-1)\ell(g(x))\}^2 z(x) g^{-\sigma}(x) L^{-1}(g(x))\,, \quad x \to \infty,$$

and denote the function on the right-hand side of (2.28) by F(x), and the inverse function of g(x) by h(x). Then, it follows from (2.28), that F(h(x)) is o-r.v.f. of index $-\sigma$ at infinity, since z(x) and h(x) are s.v.f. Also, let us put $F_1(y) = \frac{\phi(y)}{y}$, $F_2(x) = F_1(y(x))$. Then, $F_1(y)$ is o-r.v.f. at zero of index $\lambda-1$, and, due to Proposition 1.5 ,its inverse function F_1^{-1} is o-r.v.f. of index $1/(\lambda-1)$. Due to Proposition 1.7, $F_2(h(x))$ is also o-r.v.f. of index $-\sigma$. Finally, due to Proposition 1.6, the function $y(h(x)) = F_1^{-1}\{F_2(h(x))\}$ is o-r.v.f. at infinity of index $-\sigma/(\lambda-1)$, and also $L_1\{y(h(x))\}$ and z(h(x)) are s.v.f. at infinity.

Now, let us rewrite (2.28) as

(2.29) $$x^{\sigma/(\lambda-1)} f(x)\phi(y(h(x))) \sim$$
$$\sim \{\sigma^2(\lambda-1)^2 \ell^2(x) z(h(x))\}^{\lambda/(\lambda-1)} \{L(x) L_1(y(h(x)))\}^{1/(1-\lambda)},$$
$$x \to \infty \,.$$

The preceding discussion shows that the right-hand side function in (2.29) is s.v.f, which, then, also holds for the left-hand side one, say $\tilde{L}(x)$, because of Proposition 1.7 ;

thus, by integrating (2.1) over $(h(x),\infty)$,

$$-y'(h(x)) = \int_1^\infty (xt)^{-\sigma/(\lambda-1)} \frac{\tilde{L}(xt)}{t\ell(xt)}\,dt$$

or, by Proposition 1.2,

$$(2.30) \qquad -\frac{y'(x)}{\phi(y(x))} \sim \frac{\lambda-1}{\sigma}\,\frac{g(x)}{g'(x)}\,f(g(x)), \quad x\to\infty.$$

Integrating (2.30) over (a,x), and applying Proposition 1.4 to the left-hand side integral, and Proposition 1.1 to the right-hand side one, one gets

$$\frac{y(x)}{\phi(y(x))} \sim \left\{\frac{\lambda-1}{\sigma}\,\frac{g(x)}{g'(x)}\right\}^2 f(g(x)), \quad x\to\infty,$$

which is equivalent to (2.2).

3. ASYMPTOTIC ESTIMATE OF SOLUTIONS

We can extend the classes to which the functions f and ϕ belong, by using Definitions 1.1 and 1.2, but then we obtain some inequalities satisfied by the solutions of (2.1) for $x > x_o$, instead of the precise asymptotic behaviour.

First notice that $f(x) \in R_\infty(-p,q)$, due to Definitions 1.1 and 1.2, implies that there exist two numbers A_1, A_2 $(0 < A_1 < 1,\ A_2 > 1)$, such that for $x > x_o$ and $\omega > 1$

$$A_1\omega^p < \frac{f(\omega x)}{f(x)} < A_2\omega^q .$$

Analogously, $\phi(y) \in R_o(-s,r)$ implies the existence of two numbers B_1, B_2 $(0 < B_1 < 1,\ B_2 > 1)$, such that for $y\to 0$ and $\omega > 1$

$$(3.1) \qquad B_1\omega^r < \frac{\phi(\omega y)}{\phi(y)} < B_2\omega^s .$$

We shall prove

THEOREM 3.1. *Let* $f(x) \in R_\infty(-p,q)$, $0<p<q$, $\phi(y) \in R_0(-s,r)$, $1<r<s$, *with*

$$B_2(s-1) < 1 + B_1(r-1) , \tag{3.2}$$

B_i $(i=1,2)$ *defined in (3.1). Further, let* $g(x)$ *be a positive, continuously differentiable function tending to infinity with* x, *and such that* $g'(x)/g(x) = \ell(g(x))$, *where* $\ell(t)$ *is s.v.f. at infinity. Then, for any positive solution* $y(x)$ *tending to zero for* $x \to \infty$ *of equation (2.1), there holds for* $x \geq x_0$

$$m\{\frac{g'(x)}{g(x)}\}^2\{f(g(x))\}^{-1} < \frac{\phi(y(x))}{y(x)} < M\{\frac{g'(x)}{g(x)}\}^2\{f(g(x))\}^{-1} . \tag{3.3}$$

P r o o f of Theorem 3.1. The proof begins with (2.6), and follows the one of Theorem 2.1, but using inequalities for J, J' and $\frac{\phi(y)}{y}$ instead of the asymptotic formulae (2.7), (2.8), (2.10). The inequalities are obtained by using (2.18)-(2.21) and Proposition 1.9. Thus, instead of (2.11), one obtains for $x \geq x_0$

$$\begin{aligned} & m\{\frac{g(x)}{g'(x)}\}^2 f(g(x)) \frac{\phi(y(x))}{y(x)} < \\ & < z(x) < M\{\frac{g(x)}{g'(x)}\}^2 f(g(x)) \frac{\phi(y(x))}{y(x)} . \end{aligned} \tag{3.4}$$

In addition, condition (3.2) implies that there exist constants d, d_1, d_2 such that

$$0 < d_1 < v_1(y) < d_2, \quad v_2(y) < d < 1 . \tag{3.5}$$

To complete the proof, we need the following

LEMMA 3.1. *For any* $d<1$ *there exist positive numbers* α_1, α_2 *such that for* $x \geq x_0$

$$\alpha_1 \frac{g'(x)}{g(x)} < (2-d)\frac{J'(x)}{J(x)} - \frac{J''(x)}{J'(x)} < \alpha_2 \frac{g'(x)}{g(x)} .$$

If $z(x)$ oscillates for $x > x_0$, then, because of (3.5), there exist points $x_i > x_0$ $(i = 1,2,\dots)$ such that

$$z(x_i) > \{(2-d)\frac{J'}{J} - \frac{J''}{J'}\} d_2^{-1} \frac{J'}{J''}$$

and the points $x_i' > x_0$ $(i = 1,2,\dots)$ such that

$$z(x_i') < (2\frac{J'}{J} - \frac{J''}{J'}) d_1^{-1} \frac{J'}{J''} .$$

It follows by Lemma 3.1. that the minima of $z(x)$ lie above or on the line $y = c_1 > 0$, and the maxima lie under the line $y = c_2 > 0$, whence $0 < c_1 \le z(x) \le c_2$, $x > x_0$.

If, on the other hand, $z(x)$ is monotone, the proof is analogous to that of Theorem 2.1 , except that to prove that $z(x)$ cannot tend to zero, one has to use again Lemma 3.1 , with $d = 0$. Consequently, (2.17) holds, and (3.3) follows from (3.4).

Therefore, we are left with the

P r o o f of Lemma 3.1. The right-hand side inequality follows directly by using (2.18), (2.19) and Proposition 1.9. To prove the left-hand side one, suppose on the contrary that for each $\alpha_1 > 0$ and for $x_1 > x_0$ it does not hold. Then, because of the continuity of the intervening functions, one has in an interval (x, kx), containing x_1

$$(3.6) \qquad (2-d)\frac{J'(t)}{J(t)} - \frac{J''(t)}{J'(t)} < \alpha_1 \frac{g'(t)}{g(t)} .$$

By integrating (3.6) over that interval, one obtains

$$\{\frac{J(kx)}{J(x)}\}^{2-d} \frac{J'(x)}{J'(kx)} < \{\frac{g(kx)}{g(x)}\}^{\alpha_1} ,$$

from whence, due to (2.18), (2.19) and Proposition 1.9, there follows the existence of a constant $\beta > 0$, such that for all $\varepsilon > 0$

$$(3.7) \qquad \beta\{\frac{g(kx)}{g(x)}\}^{p(1-d)-\varepsilon(3-2d)} < \{\frac{g(kx)}{g(x)}\}^{\alpha_1} .$$

But, inasmuch as $g(kx)/g(x) \to \infty$, $x \to \infty$ for all $k > 1$, as is shown at the beginning of the proof of Theorem 2.1 , the inequality (3.7) cannot hold with ε and α_1 chosen in such a way that $p(1-d) - \varepsilon(3-2d) > \alpha_1$.

EXAMPLE 3.1.

$$y'' = c\, e^{\sigma x^{\alpha}} x^{\beta} (a+\sin x)\, y^{\lambda} (b + \cos y\ ),$$

$$a > 1,\quad b > 1,\quad c > 0,\quad \sigma > 0,\quad \lambda > 1,\quad 4b(\lambda-1) < b^2 - 1\ .$$

Here $g(x) = e^{x^{\alpha}}$, $f(t) = ct^{\sigma}(\ln t)^{\beta/\alpha}(a+\sin \ln^{1/\alpha} t)$, $\phi(y) = y^{\lambda}(b + \cos y)$, and the conditions of Theorem 3.1. are fulfilled so that for $x > x_o$

$$m\{e^{\sigma x^{\alpha}} x^{2(1-\alpha)+\beta} (a+\sin x)\}^{-1} < y^{\lambda-1}(x)\{b + \cos y(x)\} <$$

$$< M\{e^{\sigma x^{\alpha}} x^{2(1-\alpha)+\beta} (a+\sin x)\}^{-1}$$

or, since $b > 1$,

$$m\{e^{\sigma x^{\alpha}} x^{2(1-\alpha)+\beta} (a+\sin x)\}^{1/(1-\lambda)} < y(x) <$$

$$< M\{e^{\sigma x^{\alpha}} x^{2(1-\alpha)+\beta} (a+\sin x)\}^{1/(1-\lambda)}\ .$$

However Theorem 2.1. cannot be applied, since $f(t)$ is not o-r.v.f. at ∞ , and also $\phi(y)$ is not o-r.v.f. at zero.

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REZIME

ASIMPTOTIKA REŠENJA JEDNE KLASE NELINEARNIH DIFERENCIJALNIH JEDNAČINA DRUGOG REDA

Razmatrana je asimptotika rešenja koja teže nuli kad $x \to \infty$ jednačine $y'' = f(g(x))\phi(y)$, gde je $f(g(x))$ funkcija neograničenog rasta.