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APPLICATIONS OF THE S-ASYMPTOTIC

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ABSTRACT

The relation between the S-asymptotic and the mappings: (E') into (D') and (D') into (D') is given. We proved a theorem with the necessary and sufficient conditions that a linear partial differential equation has a solution with the given S-asymptotic.

INTRODUCTION

There are many definitions of the asymptotic behaviour of a distribution at infinity; we shall cite only two [2], [3]. But we shall use here the S-asymptotic [4] inspired by notions of L. Schwartz [6] T. II p. 97 and [1] p. 44. The S-asymptotic can be applied in many cases. We shall use it here especially to find solutions of a convolution equation which have prescribed behaviour at infinity.

1. S-ASYMPTOTIC

By (D') we shall denote the set of Schwartz distributi-

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ons, by (S′) the set of tempered distributions and by (E′) the set of distributions with the compact support.

Let Γ' be a cone in R^n; $\Sigma(\Gamma')$ is the set of numerical functions which maps Γ' into R in such a way that if $c(h) \in \Sigma(\Gamma')$, then $c(h) \neq 0$, $\|h\| \geq \beta_c$.

DEFINITION 1. *A distribution* $T \in (D')$ *has the S-asymptotic in the cone* $\Gamma \subset \Gamma'$ *related to the function* $c(h) \in \Sigma(\Gamma')$ *and with the limit* $U \in (D')$, *if there exists*

$$(1) \qquad \lim_{h\in\Gamma, \|h\|\to\infty} \langle T(t+h)/c(h), \varphi(t) \rangle = \langle U, \varphi \rangle, \quad \varphi \in (D).$$

In this case we write $T(t+h) \overset{S}{\sim} c(h)U(t)$, $h \in \Gamma$.

The S-asymptotic generalizes the asymptotic of numerical functions and retains some of its properties. Such an one is the following.

PROPERTY 1. *If for every* $r > 0$, *there exists* $\beta_r \in R_+$, *such that the sets* $\{t \in R^n,\ t \in (\operatorname{supp} T - h) \cap B(0,r),\ \|h\| \geq \beta_r,\ h \in \Gamma\}$ *are empty, then* $T(t+h) \overset{S}{\sim} c(h)\cdot 0$, $h \in \Gamma$ *for every* $c(h) \in \Sigma(\Gamma')$. ($B(0,r) = \{t \in R^n, \|t\| \leq r\}$).

For some special cones we can characterize the limit U and the numerical function c(h). Thus, if the cone Γ is the ray $\{\beta w, \beta > 0, \|w\| = 1\}$ we have the following theorem [4]:

THEOREM A. *Let* $T \in (D')$ *and* $T(t+\beta w) \overset{S}{\sim} c(\beta)U(t)$, $\beta \in R_+$; $U \neq 0$. *Then:*

a) There exists $\lim_{\beta\to\infty} c(\beta_0+\beta)/c(\beta) = d(\beta_0) < \infty, \beta_0 \in R$.

b) The limit U *satisfies the equation* $U(t+\beta w) = d(\beta)U(t)$, $t \in R^n$, $\beta \in R$.

c) $d(\beta) = \exp(\alpha\beta)$, $\beta \in R$, *where* $\alpha = d'(0)$.

d) If $w_i \neq 0$ *for* $i = k_1, \ldots, k_m$, *then* $U(t) = V(t)\exp\cdot\left(\frac{\alpha}{m}\sum_{i=k_1}^{k_m}\frac{t_i}{w_i}\right)$ *where* V(t) *is a solution of the equation*

$$\sum_{i=k_1}^{k_m} w_i \partial V(t)/\partial t_i = 0.$$

2. S-ASYMPTOTIC AND MAPPINGS

PROPOSITION 1. *Let* $S \in (\mathcal{E}')$ *and* $T \in (D')$. *If* $T(t+h) \overset{S}{\sim} c(h)U(t)$, $h \in \Gamma$, *then* $(S*T)(t+h) \overset{S}{\sim} c(h)(S*U)(t)$, $h \in \Gamma$.

PROOF. We know that $\delta_{-h}*(S*T) = S*(\delta_{-h}*T)$ ([6] Ch. VI, Th. 7). Hence $(S*T)(t+h) = S*T(t+h)$. The mapping $(S,T) \to S*T$ which maps $(\mathcal{E}')\times(D')$ into (D') is continuous in both variables ([6] Ch. VI, Th. V), which ends the proof.

CONSEQUENCES OF PROPOSITION 1. 1. If we take for $S = \delta^{(k)}$, $k = (k_1,\dots,k_n)$, Proposition 1 says: From $T(t+h) \overset{S}{\sim} c(h)U^{(k)}(t)$, $h \in \Gamma$ follows $T^{(k)}(t+h) \overset{S}{\sim} c(h)U^{(k)}(t), h\in\Gamma$.

2. For a convolution equation

$$S * X = T, \quad S \in (\mathcal{E}'), \quad T \in (D') \tag{2}$$

a necessary condition that a solution X of equation (2) has the S-asymptotic in the cone Γ, related to the function $c(h) \in \Sigma(\Gamma')$ and the limit $U \in (D')$, is that T has the same S-asymptotic with the limit $S * U$.

3. Let us suppose that T has Property 1. If X has an S-asymptotic with the limit U, then U is the solution of the equation $S * U = 0$.

PROPOSITION 2. *Let us suppose that the mapping L which maps* $(\mathcal{E}')$ *into* (D') *has the following properties: It is linear, continuous and keeps the translation* $((Lf)(t+h) = Lf(t+h))$. *A necessary and sufficient condition that L maps* $(\mathcal{E}')$ *into the set* $\{T \in (D'),\ T(t+h) \overset{S}{\sim} c(h)U_T(t),\ h \in \Gamma\}$ *is that there exists* $V\in(D')$ *such that*

$$(L\delta)(t+h) \overset{S}{\sim} c(h)V(t), \quad h \in \Gamma \tag{3}$$

In this case for $S \in (E')$ $(LS)(t+h) \overset{S}{\sim} c(h)(S*V)(t)$, $h \in \Gamma$.

PROOF. The condition is necessary. We know that $L\ldots = f_0 * \ldots$, where $f_0 \in (D')$ ([7], p. 81). As $\delta \in (E')$, then $(L\delta)(t+h) \overset{S}{\sim} c(h)U_\delta(t)$, $h \in \Gamma$. And this is our condition (3).

Condition (3) is sufficient. $LS = f_0 * S$ and $L\delta = f_0 * \delta = f_0$. Hence, condition (3) says that $f_0(t+h) \overset{S}{\sim} c(h)V(t)$, $h \in \Gamma$. Now $(LS)(t+h) = (f_0 * S)(t+h) = S * f_0(t+h)$. By Proposition 1 we have the statement of Proposition 2.

PROPOSITION 3. *Let us suppose that the mapping* M *which maps* (D') *into* (D') *has the following properties: It is linear, continuous and keeps the translation. If* $T(t+h) \overset{S}{\sim} c(h)U(t)$, $h \in \Gamma$, *then* $(MT)(t+h) \overset{S}{\sim} c(h)(M\delta)*U$, $h \in \Gamma$.

PROOF. We know that $M\ldots = g_0 * \ldots$, where $g_0 \in (E')$. $(MT)(t+h) = g_0 * T(t+h) = (M\delta)*T(t+h)$. There remains only to use our Proposition 1.

3. S-ASYMPTOTIC OF THE SOLUTIONS OF A LINEAR PARTIAL DIFFERENTIAL EQUATION

In this third part we shall consider the S-asymptotic in (S'). (S' is the set of tempered distributions). In this case $T(t+h) \overset{S}{\sim} c(h)U(t)$, $h \in \Gamma$ means that the limit

$$\lim_{h \in \Gamma, \|h\| \to \infty} T(t+h)/c(h) = U(t)$$

exists in (S').

For the occasions the S-asymptotic in (S') follows from the S-asymptotic in (D') see [5]. The following theorem concerns the behaviour of tempered distributions at infinity as elements of (D').

PROPOSITION 4. *If* $\lim_{h \in R^n, \|h\| \to \infty} \|h\|^k / c(h) = 0$ *for every*

$k \in N$, *then for every* $T \in (S')$, $T(t+h) \overset{S}{\sim} c(h)\cdot 0$, $h \in \Gamma$ *in* (D').

PROOF. Let us suppose that $T \in (S')$. Then there exists $k > 0$ such that the set $\{T(t+h)/(1+\|h\|^2)^{k/2},\ \|h\| \geq \beta_0\}$ is bounded in (D') ([6] T.II, p. 95). Now for $\varphi \in (D)$:

$$\lim_{h\in\Gamma,\|h\|\to\infty} < T(t+h)/c(h),\ \varphi(t) >$$

$$= \lim_{h\in\Gamma,\|h\|\to\infty} \frac{(1+\|h\|^2)^{k/2}}{c(h)} < \frac{T(t+h)}{(1+\|h\|^2)^{k/2}},\ \varphi(t) >$$

$$= < U(t),\ \varphi(t) >,$$

which gives that $U = 0$.

In the following we shall denote by F and F^{-1} the Fourier transform and its inverse.

PROPOSITION 5. *Let* $g \in (S')$ *and* $f = F[g]$. *A necessary and sufficient condition that there exists*

$$(4) \qquad \lim_{h\in\Gamma,\|h\|\to\infty} g(t+h)/c(h) = U(t) \text{ in } (S')$$

is the existence of the limit

$$(5) \qquad \lim_{h\in\Gamma,\|h\|\to\infty} \frac{1}{c(h)} \exp(-i<t,h>)f(t) = V(t) \text{ in } (S'),$$

and in this case $U(t) = F^{-1}[V](t)$.

PROOF. If $g(x) = F^{-1}[f](x)$, then $g(x+h) = F^{-1}[\exp(-i\cdot<t,h>)f(t)](x)$ and

$$(6) \qquad < g(x+h)/c(h),\ \phi(t) > = < \frac{1}{c(h)} F^{-1}[\exp(-i<t,h>)\cdot f(t)](x),\ \phi(x) >.$$

If $\frac{1}{c(h)}\exp(-i<t,h>)f(t)$ converges in (S′) to V(t) when $\|h\| \to \infty$, $h \in \Gamma$, because of the continuity of the Fourier transform, it follows that

$$\lim_{h\in\Gamma,\|h\|\to\infty} \langle g(x+h)/c(h),\phi(x)\rangle = \langle F^{-1}[V](x),\phi(x)\rangle.$$

Let us suppose now that limit (4) exists in (S′) then there exists the limit

$$\lim_{h\in\Gamma,\|h\|\to\infty} \frac{1}{c(h)} F^{-1}[\exp(-i<t,h>)f(t)](x) \quad \text{in } (S').$$

We know that

$$\exp(-i<t,h>)f(t) = F\,[F^{-1}[\exp(-i\langle y,h\rangle)f(y)](x)](t).$$

Because of the continuity of operation F^{-1}, there follows the statement of Proposition 5.

Now we can pass on to the partial differential equations. First, we shall introduce the following notations:

Let P(y), $y \in R^n$ be a polynomial. By reg 1/P(y) we denote a solution, belonging to (S′), of the equation $P(y)\cdot X = 1$. It is well known that L. Hörmander proved that the last equation can always be solved in (S′) if $P(y) \neq 0$.

PROPOSITION 6. *A necessary and sufficient condition that there exists a solution to the equation*

(7) $$L(D)E = \delta,\quad L(D) = \sum_{|\alpha|\geq 0}^{m} a_\alpha D^\alpha,\quad a_\alpha \in R,\quad \alpha \in N_0^n.$$

such that

(8) $$E(t+h) \overset{S}{\sim} c(h)U(t),\quad h \in \Gamma \text{ in } (S')$$

is that there exists

$$(9) \qquad \lim_{h\in\Gamma,\|h\|\to\infty} \frac{1}{c(h)} \exp(-i\langle t,h\rangle) \operatorname{reg} \frac{1}{L(-it)} = F[U](t)$$

in (S').

PROOF. We know that $E \in (S')$ is a fundamental solution of operator $L(D)$ if and only if $F[E]$ is a solution to equation $L(-ix)\, F[E] = 1$ (see [8], p. 192). There remains only to apply our Proposition 5.

THEOREM. *A necessary and sufficient condition that there exists a solution* X *to the equation*

$$(10) \qquad L(D)X = G, \quad G \in (E')$$

such that $X(t+h) \overset{S}{\sim} c(h)\ (G * U)(t),\ h \in \Gamma$ in (S') *is that there exists limit (9).*

PROOF. The existence of limit (9) is sufficient. From Proposition 6 it follows that limit (9) is necessary and sufficient for $E(t+h) \overset{S}{\sim} c(h)U(t),\ h \in \Gamma$ in (S') where E is a solution of equation (10). To find the S-asymptotic of X we have only to apply Proposition 1.

Limit (9) is necessary. Let us suppose that there exists a solution X of equation (10) such that $X(t+h) \overset{S}{\sim} c(h)\,\cdot$ $\cdot\,(G * U)(t),\ h \in \Gamma$. We know that every solution X of equation (10) has the form $X = G * E$, where E is a solution of equation (7) ([8] p. 194) and $F[E] = \operatorname{reg}[1/L(-ix)]$. By Proposition 6 there exists the limit

$$\lim_{h\in\Gamma,\|h\|\to\infty} \frac{1}{c(h)} \exp(-i\langle t,h\rangle)\, F[G]\cdot F[E] = F[G]\cdot F[U],$$

hence follows relation (9).

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REZIME

PRIMENA S-ASIMPTOTIKA

Obeležimo sa (D') prostor distribucija L. Schwartz-a, a sa (S') i (E') prostore distribucija sporoga rasta, odnosno prostor distribucija sa kompaktnim nosačem. Γ' je konus u R^n, a Σ(Γ') je skup numeričkih funkcija koje preslikavaju Γ' u R, takvih da je za c(h) ∈ Σ(Γ') c(h) ≠ 0, ‖h‖ ⩾ β_c.

Koristimo se sledećom definicijom S-asimptotike [4]:

DEFINICIJA. *Distribucija* T ∈ (D') *ima S-asimptotiku u konusu* Γ ⊂ Γ' *u odnosu na funkciju* c(h) ∈ Σ(Γ') *i granicu*

$U \in (D')$ *ako postoji*

(1) $$\lim_{h\in\Gamma, \|h\|\to\infty} \langle T(t+h)/c(h), \varphi(t) \rangle = \langle U, \varphi \rangle, \ \varphi\in(D).$$

To ćemo skraćeno pisati: $T(t+h) \overset{S}{\sim} c(h)U(t), \ h \in \Gamma$.

U prvom delu ukazano je na već poznate karakteristične osobine S-asimptotike. U drugom delu pokazan je odnos izmedju S-asimptotike i : konvolucija kada ona preslikava $(E') \times (D')$ u (D'), linearnog preslikavanja koje preslikava (E') u (D') kao i linearnog preslikavanja koje preslikava (D') u (D').

U trećem delu, prvo dajemo karakteristiku elemenata iz (S') kao podskupa (D') pomoću S-asimptotike. Zatim pokazujemo odnos S-asimptotike i Fourierove transformacije. Najzad, dokazujemo teoremu za parcijalne diferencijalne jednačine:

TEOREMA. *Potreban i dovoljan uslov da postoji rešenje* X *jednačine (10) koje ima S-asimptotiku u* (S'):

$$X(t+h) \overset{S}{\sim} c(h)(G*U)(t), \quad h \in \Gamma$$

je da postoji granica data u relaciji (9).