ZBORNIK RADOVA
Prirodno-matematičkog fakulteta
Univerziteta u Novom Sadu
Serija za matematiku, 14,2 (1984)

REVIEW OF RESEARCH
Faculty of Science
University of Novi Sad
Mathematics Series, 14, 2 (1984)

SUBSETS AND PARACOMPACTNESS

Ilija Kovačević
Fakultet tehničkih nauka
21000 Novi Sad, ul. V. Vlahovića br.3, Jugoslavija

ABSTRACT

The aim of the present paper is to study some properties of α-paracompact sets of a topological space which is not Hausdorff or regular. It will be shown that some properties of α-paracompact sets of a Hausdorff or regular space are valid although the topological space is not Hausdorff or regular. Notation is standard except that $\alpha(A)$ will be used to denote the interior of the closure of A.

1. PRELIMINARIES

Throughout the present paper, space will always mean a topological space on which no separation axioms are assumed, unless explicitly stated.

DEFINITION 1.1. *Let* X *be a space and* A *a subset of* X. *The set* A *is* <u>*α-paracompact*</u> *iff every* X-*open cover of* A *has an* X-*open* X-*locally finite refinement which covers* A, [4].

DEFINITION 1.2. *A subset* A *of a space* X *is* <u>*regu-*</u>

AMS Mathematics Subject Classification (1980): 54D18.
Key words and phrases: α-paracompact, α-Hausdorff, α-regular.

larly open iff it is the interior of its own closure, or equivalently, iff it is the interior of some closed set. A *is called regularly closed iff it is the closure of its own interior, or equivalently, iff it is the closure of some open set (a subset is regularly open iff its complement is regularly closed)*, [3].

DEFINITION 1.3. *A mapping* $f : X \to Y$ *is said to be almost closed (almost open) iff for every regularly closed (regularly open) set* F *of* X, f(F) *is closed (open) in* Y, [3].

2. ON α-PARACOMPACT SETS

DEFINITION 2.1. *A subset* A *of a space* X *is α-Hausdorff iff any two points* a,b *of a space* X, *where* $a \in A$ *and* $b \in X \setminus A$, *can be strongly separated.*

THEOREM 2.1. *If* A *is an α-Hausdorff α-paracompact subset of a space* X *and* x *is a point of* $X \setminus A$, *then there are disjoint regularly open neighbourhoods of* x *and* A. *Consequently, each α-Hausdorff α-paracompact subset of a space* X *is closed.*

PROOF: Let A be any α-Hausdorff α-paracompact subset of a space X, and x be any point of X\A. Since A is α-Hausdorff, then for each point $a \in A$, there exist disjoint open sets U_a and V_a such that

$$a \in U_a, \quad x \in V_a .$$

Then,

$$\mathcal{U} = \{U_a : a \in A\}$$

is an X-open covering of A, hence there exists an X-locally finite X-open family

$$\mathcal{H} = \{H_i : i \in I\}$$

which refines $\mathcal{U}$ and covers A. Since $\mathcal{H}$ is X-locally finite, then there exists an open set M containing x such that M intersects finitely many members of H. Let $I_o \subset I$ is a finite subset of a set I such that $M \cap H_i \neq \emptyset$ for each $i \in I_o$ and $M \cap H_i = \emptyset$ for each $i \in I \setminus I_o$. For each $i \in I$ there exists $a_i \in A$ such that $H_i \subset U_{a_i}$. Let

$$U = \cup \{H_i : i \in I\} \quad \text{and} \quad V_x = M \cap (\cap \{V_{a_i} : i \in I_o\}) .$$

Then, U and V_x are open disjoint neighbourhoods of A and x, respectively, hence $\alpha(U)$ and $\alpha(V_x)$ are regularly open disjoint neighbourhoods of A and x, respectively.

COROLLARY 2.1. ([2]) *Every α-paracompact subset of a Hausdorff space is closed.*

We know that Theorem 2.1 is true, when a space X is Hausdorff (every subset of a Hausdorff space is α-Hausdorff). The following example shows that there exists an α-Hausdorff α-paracompact subset of a space which is not Hausdorff.

EXAMPLE 2.1. Let

$$X = \{a_i, b_i : i = 1,2,\ldots\} .$$

Let

$$A = \{a_i : i = 1,2,\ldots\} .$$

Let each point b_i be isolated. For each point $a \in A$ let the fundamental system of neighbourhoods of a be the set A. The set A is α-Hausdorff α-paracompact, but X is not Hausdorff (for two points of the subset A there are no disjoint open neighbourhoods).

THEOREM 2.2. *For any two disjoint subset A and B of a space X, where A is α-Hausdorff α-paracompact and B α-paracompact, there exist disjoint regularly open neighbourhoods of A and B respectively.*

PROOF: For each point $x \in B$ there exist disjoint open sets U_x and V_x such that

$$x \in U_x, \quad A \subset V_x .$$

The family

$$\mathcal{U} = \{U_x : x \in B\}$$

is an X-open covering of the α-paracompact subset B, hence there exists an X-open X-locally finite family

$$\mathcal{H} = \{H_i : i \in I\}$$

which refines $\mathcal{U}$ and covers B.

Let

$$H = \bigcup \{H_i : i \in I\} .$$

Then

$$B \subset H, \quad \overline{H} = \bigcup \{\overline{H}_i : i \in I\} .$$

For each $i \in I$ there exists $x_i \in B$ such that $H_i \subset U_{x_i}$. Since $A \subset V_{x_i}$, it follows that for each $i \in I$

$$A \cap \overline{H}_i = \emptyset, \text{ i.e. } A \subset X \setminus \overline{H}_i$$

($U_{x_i} \cap V_{x_i} = \emptyset$ implies that $\overline{H}_i \cap V_{x_i} = \emptyset$). Hence we have

$$A \subset X \setminus \overline{H} = U .$$

U and H are disjoint open sets such that $A \subset U$ and $B \subset H$, hence $\alpha(U)$ and $\alpha(H)$ are disjoint regularly open sets such that $A \subset \alpha(U)$ and $B \subset \alpha(V)$.

COROLLARY 2.2. ([1]) *Every two disjoint α-paracompact subsets of a Hausdorff space can be strongly separated.*

Using a similar method as in [1], we shall prove the following theorem.

THEOREM 2.3. *Let* $f : X \to Y$ *be an almost closed mapping of a space* X *onto a space* Y, *such that* $f^{-1}(y)$ *is α-Hausdorff α-paracompact for each point* $y \in Y$, *then* Y *is Hausdorff.*

PROOF: Let y_1 and y_2 be any distinct points of Y. Then, $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are disjoint α-Hausdorff α-paracomcompact subsets of X. Then, by the preceding Theorem, there exist disjoint regularly open sets U_1 and U_2 such that $f^{-1}(y_1) \subset U_1$ and $f^{-1}(y_2) \subset U_2$. Since f is almost closed, then there exist open sets V_1 and V_2 containing y_1 and y_2, respectively, such that

$$f^{-1}(V_1) \subset U_1, \quad f^{-1}(V_2) \subset U_2 .$$

Hence the result.

COROLLARY 2.3 ([1]) *Let* $f : X \to Y$ *be an almost closed mapping of a Hausdorff space* X *onto a space* Y *such that* $f^{-1}(y)$ *is α-paracompact for each point* $y \in Y$, *then* Y *is Hausdorff.*

EXAMPLE 2.2. Let

$$X = \{a_i, b_i : i = 1,2,3,\dots\} .$$

Let

$$A = \{a_i : i = 1,2,\dots\}$$

be the fundamental system of neighbourhoods of a_i and let the fundamental system of neighbourhoods of b_i be the set

$$B = \{b_i : i = 1,2,\dots\} .$$

Let

$$Y = \{a,b\} \quad \text{and} \quad \tau_Y = \{\emptyset, \{a\}, \{b\}, Y\} .$$

Let $f : X \to Y$ be a mapping of a space X onto a space Y defined by

$$f(a_i) = a, \quad f(b_i) = b, \quad i = 1,2,\dots$$

The mapping f is almost closed, such that $f^{-1}(y)$ is α-Hausdorff α-paracompact, but X is not Hausdorff.

DEFINITION 2.2. *A subset* A *of a space* X *is α-regular iff for any point* $a \in A$ *and any* X*-open set* U *containing* a, *there exists an* X*-open set* V *such that*

$$a \in V \subset \overline{V} \subset U$$

or equivalently, for any closed set F *of a space* X *and any point* $x \in A$ *such that* $x \in X\setminus F$, *there exist disjoint open neighbourhoods of* x *and* F, *respectively.*

THEOREM 2.4. *If* A *is an* α*-regular* α*-paracompact subset of a space* X, *then* $\overline{A}$ *is* α*-paracompact.*

PROOF: Let

$$\mathcal{U} = \{U_i : i \in I\}$$

be any X-open covering of $\overline{A}$. For each $x \in A$, there exists U_i containing x. Since A is α-regular, there exists an open set V_x such that

$$x \in V_x \subset \overline{V}_x \subset U_i \ .$$

Consider the open covering

$$\mathcal{V} = \{V_x : x \in A\}$$

of the set A.
Since A is α-paracompact, there exists an X-locally finite family of X-open sets

$$\mathcal{W} = \{W_j : j \in J\}$$

which refines $\mathcal{V}$ and covers A.
For each $x \in A$, there exists W_j such that $x \in W_j$. Since A is α-regular there exists an open set B_x such that

$$x \in B_x \subset \overline{B}_x \subset W_j \ .$$

Since

$$\{B_x : x \in A\}$$

is an open covering of the α-paracompact subset A, there exist an X-locally finite X-open refinement

$$\{H_k : k \in K\}$$

of $\{B_x : x \in A\}$ which covers A. Then,

$$\bar{A} \subset \bigcup \{\overline{H_k : k \in K}\} = \bigcup \{\bar{H}_k : k \in K\} .$$

$H_k \subset B_{x(k)}$ for some $x(k) \in A$, i.e. $\bar{H}_k \subset \bar{B}_{x(k)} \subset W_{j_o}$ for some $j_o \in J$.

Thus

$$\{W_j : j \in J\}$$

is an X-locally finite X-open refinment of the open family $\{U_i : i \in I\}$ such that

$$\bar{A} \subset \bigcup \{W_j : j \in J\} ,$$

hence $\bar{A}$ is α-paracompact.

COROLLARY 2.4. *If A is any α-paracompact subset of a regular space X, then $\bar{A}$ is α-paracompact.*

The closure of an α-regular α-paracompact is not always α-regular. The following example serves the purpose .

EXAMPLE 2.3. Let

$$X = \{a,b,c_i,a_i : i = 1,2,\dots\} .$$

Let

$$A = \{b,a_i : i = 1,2,\dots\} .$$

Let each point a_i and c_i be isolated. Let the fundamental system of neighbourhoods of a be the set

$$\{V^n(a) : n = 1,2,\dots\}$$

where

$$V^n(a) = \{a,a_i : i > n\} .$$

Let the fundamental system of neighbourhoods of b be the set

$$\{U^n(b) : n = 1,2,\dots\}$$

where

$$U^n(b) = \{b,a,a_i : i > n\} .$$

The set A is α-regular α-paracompact.

$$\overline{A} = \{b, a, a_i : i = 1,2,\ldots\}$$

is α-paracompact, but is not α-regular (for any open neighbourhood $V^n(a)$ of a, $\overline{V^n(a)} = V^n(a) \cup \{b\}$).
X is not regular at the point a, hence X is not regular.

THEOREM 2.5. *If A is an α-regular α-paracompact subset of a space X, U an open neighbourhood of A, then there exists an open neighbourhood V of A such that*

$$A \subset V \subset \overline{V} \subset U .$$

PROOF: For each point $x \in A$ there exists an open set W_x such that

$$x \in W_x \subset \overline{W}_x \subset U .$$

Now, the family

$$\mathcal{W} = \{W_x : x \in A\}$$

is an X-open covering of A, hence there exists an X-locally finite X-open family $\mathcal{V}$, which refines $\mathcal{V}$ and covers A. Let

$$V = \bigcup \{V_i : V_i \in \mathcal{V}\} .$$

Then

$$A \subset V \subset \overline{V} = \overline{\bigcup \{V_i : V_i \in \mathcal{V}\}} = \bigcup \{\overline{V}_i : V_i \in \mathcal{V}\} \subset U .$$

REFERENCES

[1] Kovačević, I., *A note on α-nearly paracompact (α-almost paracompact) sets and almost closed mappings (to appear).*

[2] Kovačević, I., *Locally nearly paracompact spaces, Publ. Inst. Math. (N.S.)(Beograd), 29(43), (1981), 117-124.*

[3] Singal, M.K. and Singal, A.R., *Almost continuous mappings, Yokohoma Math. J., 16 (1968), 63-73.*

[4] Wine, J.D., *Locally paracompact spaces*, *Glasnik matematički*, 10(30) (1975), 351-357.

Received by the editors January 23,1985.

REZIME

PODSKUPOVI I PARAKOMPAKTNOST

U radu se ispituju neke osobine α-parakompaktnih podskupova u topološkim prostorima koji nisu ni Hausdorffovi ni regularni. Daje se definicija α-Hausdorffovog podskupa kao i α-regularnog skupa.