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GENERALIZED PRESEMINORMED SPACES OF LINEAR MANIFOLDS AND BOUNDED LINEAR RELATIONS

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ABSTRACT

Our main purpose here is to extend standard theorems about the completeness of various spaces of bounded linear functions to linear relations. For this, a straightforward generalization of topological vector spaces has to be considered.

INTRODUCTION

In this paper, generalized preseminormed spaces (see Definitions 1.1, 2.1 and 2.5) are introduced and studied. The main emphasis is put on bounded linear functions from one generalized preseminormed space into another and on their generalized preseminormed spaces.

The results obtained extend well-known standard results of functional analysis and can be applied to linear relations too. An interesting consequence of the main results can be formulated as follows:

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Let $\hat{B}$ be the family of all closed-valued, lower semicontinuous, linear relations from a normed space X into a Banach space Y. For S,T $\in \hat{B}$ and a scalar λ , define

$$(S \hat{+} T)(x) = \overline{S(x) + T(x)}, \qquad (\lambda S)(x) = S(\lambda x)$$

and

$$\|S\| = \sup_{\|x\| \leq 1} \inf_{y \in S(x)} \|y\| .$$

Then $\hat{B}$ is a generalized Banach space.

The paper is almost self-contained, its reading needs only a minimum of algebra and topology, and a few things from our former papers [6],[8] and [9].Its particular case, when only preseminormed spaces and their linear functions are involved, can be taught at regular courses on functional analysis.

1. GENERALIZED VECTOR SPACES

DEFINITION 1.1. *An ordered triple* X(+,·) = (X,+,·), *where* X *is a nonvoid set and* + *and* · *are functions from* X × X *and* K × X *into* X, *whose values at* (x,y) *and* (λ,x) *are denoted by* x + y *and* λx, respectively, *will be called a generalized vector space over* K (= R *or* C) *if*

1. $(x+y) + z = x + (y+z)$, 2. $(\lambda\mu)x = \lambda(\mu x)$,

3. $\lambda(x+y) = \lambda x + \lambda y$, 4. $(\lambda+\mu)x = \lambda x + \mu x$,

5. $x + y = y + x$, 6. $1 x = x$

for all x,y,z ∈ X *and* λ,μ ∈ K.

REMARK 1.2. If X is a generalized vector space, then we write $-x = (-1)x$ for every x ∈ X.

REMARK 1.3. If X is a generalized vector space and there exists an element $0 \in X$ ($\infty \in X$) such that $x + 0 = x$ ($x + \infty = \infty$) for all $x \in X$, then we say that X has a zero (an infinity), or that X is a generalized vector space with zero (infinity).

Note that in this case 0 (∞) is unique and $\lambda 0 = 0$ ($\lambda\infty = \infty$) for all $\lambda \in K$, since we have $00 = 00 + 0 = 00 + 10 = (0 + 1)0 = 10 = 0$ ($0\infty = (-1 + 1)\infty = (-1)\infty + 1\infty = -\infty + \infty = \infty$).

REMARK 1.4. If X is a generalized vector space over K without zero (infinity) and $X_0 = X \cup \{0\}$ ($X_\infty = X \cup \{\infty\}$), where $0 \notin X$ ($\infty \notin X$), then by defining $x + 0 = 0 + x = x$ ($x + \infty = \infty + x = \infty$) and $\lambda 0 = 0$ ($\lambda\infty = \infty$) for all $x \in X_0$ ($x \in X_\infty$) and $\lambda \in K$, X_0 (X_∞) becomes a generalized vector space over K with zero (infinity).

Therefore it is not a severe restriction to suppose that a generalized vector space has a zero and an infinity. Note that a nontrivial vector space cannot have an infinity.

THEOREM 1.5. *Let X be a generalized vector space over K and $\Theta \subset X \times X$ such that*

$$\Theta(x) = \{y \in X\colon y + 0x = y\}$$

for all $x \in X$. Then Θ is a preordering on X such that

$$\Theta(x) + \Theta(y) \subset \Theta(x + y) \text{ and } \lambda\Theta(x) \subset \Theta(\mu x)$$

for all $x, y \in X$ and $\lambda, \mu \in K$. Moreover, $\Theta(x)$ is a generalized vector space K with zero for every $x \in X$.

PROOF. Simple computation.

REMARK 1.6. Note that the relation Θ^{-1} has the same properties except that the generalized vector space $\Theta^{-1}(x)$ need not have a zero.

THEOREM 1.7. *Let* X *be a generalized vector space over* K *and* $E = \Theta \cap \Theta^{-1}$. *Then* E *is an equivalence relation on* X *such that*

$$E(x) + E(y) \subset E(x+y) \quad \textit{and} \quad \lambda E(x) \subset E(\mu x)$$

for all $x, y \in X$ *and* $\lambda, \mu \in K$. *Moreover,*

$$E(x) = x + \Theta^{-1}(x) = \{y \in X : 0x = 0y\},$$

and $E(x)$ *is a vector space over* K *for every* $x \in X$.

PROOF. Left to the reader. (To check that the generalized vector space $E(x)$ is actually a vector space, note that $0x$ is the zero element of $E(x)$ and $y - y = 0y = 0x$ for all $y \in E(x)$.)

REMARK 1.8. The relations Θ and E will be called the canonical preordering and the canonical equivalence on X, respectively.

The quotient set X/E, which can also be made into a generalized vector space in a natural way, will be called the canonical decomposition of X.

Note that by using the canonical decomposition, each generalized vector space can be viewed as a union of disjoint vector spaces.

REMARK 1.9. It is worth mentioning that if X is a generalized vector space, then

(i) X has a zero if and only if $\Theta(x) = X$ for some $x \in X$, or equivalently $\bigcap_{x \in X} \Theta^{-1}(x) \neq \emptyset$;

(ii) X is a vector space if and only if $E(x) = X$ for some $x \in X$, or equivalently $E = X \times X$.

EXAMPLE 1.10. Let $\dot{X}$ be the family of all linear manifolds in a vector space X over K. For $x + M$, $y + N \in \dot{X}$ and $\lambda \in K$, define $(x + M) + (y + N) = (x + y) + (M + N)$ and $\lambda(x + M) = \lambda x + M$. Then $\dot{X}$ is a generalized vector space over R with zero and infinity such that

$$\theta(x + M) = \{y + N \in \dot{X}: M \subset N\},$$

for all $x + M \in \dot{X}$.

REMARK 1.11. Note that we have

$$E(x + M) = X/E$$

for any $x + M \in \dot{X}$, and X can be embedded into $\dot{X}$ such that

$$X = X/\{0\} = E(\{0\}).$$

EXAMPLE 1.12. Let $L = L(X,Y)$ be the family of all linear relations from one vector space X into another Y over K [6]. For $S,T \in L$ and $\lambda \in K$, define the relations $S + T$ and λS from X into Y by

$$(S + T)(x) = S(x) + T(x) \text{ and } (\lambda S)(x) = S(\lambda x).$$

Then L is a generalized vector space over K with zero and infinity such that

$$\theta(S) = \{T \in L: S(0) \subset T(0)\}$$

for all $S \in L$.

EXAMPLE 1.13. Let $L = L(X,Y)$ be the family of all functions f from one generalized vector space X into another Y over K which are linear in the sense that

$$f(x + y) = f(x) + f(y) \text{ and } f(\lambda x) = \lambda f(x)$$

for all $x,y \in X$ and $\lambda \in K$. For $f,g \in L$ and $\lambda \in K$, define the functions $f + g$ and λf on X by

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (\lambda f)(x) = \lambda f(x).$$

Then L is a generalized vector space over K such that

$$\Gamma(f) = \{g \in L : g \subset \Gamma \circ f\}$$

for any $\Gamma \in \{\Theta, E, \Theta^{-1}\}$ and $f \in L$.

REMARK 1.14. Note that if $f \in L(X,Y)$ and $\Gamma \in \{\Theta, E, \Theta^{-1}\}$ then $f \circ \Gamma \subset \Gamma \circ f$.

Note also that if X is, in addition, a vector space, then for any $g \in L(X,Y)$ we have $g \subset \Gamma \circ f$ if and only if $g(0) \in \Gamma(f(0))$.

The importance of Example 1.13 lies mainly in the following reduction principle.

THEOREM 1.15. *For* $S \in L(X,Y)$, *define the function* φ_S *on* X *by*

$$\varphi_S(x) = S(x).$$

Then the mapping

$$S \to \varphi_S$$

is a linear injection from $L(X,Y)$ *onto* $L(X,\dot{Y})$.

PROOF. Straightforward computation.

2. GENERALIZED PRESEMINORMED SPACES

DEFINITION 2.1. A real-valued function p on a generalized vector space X will be called a preseminorm on X if

(i) $\lim_{\lambda \to 0} p(\lambda x) = 0$ for all $x \in X$;

(ii) $p(\lambda x) \leq p(x)$ for all $|\lambda| \leq 1$ and $x \in X$;

(iii) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

REMARK 2.2. A preseminorm p on X will be called a semi norm if $p(\lambda x) \leq |\lambda| p(x)$ for all $\lambda \in K$ and $x \in X$.

THEOREM 2.3. *Let* p *be a preseminorm on* X *and* $x, y \in X$. *Then*

1. $p(0x) = 0$, 2. $p(\lambda x) \leq p(\mu x)$ *for all* $|\lambda| \leq |\mu|$,

3. $p(x) \geq 0$, 4. $p(\lambda x) = p(x)$ *for all* $|\lambda| = 1$,

5. $p(nx) \leq np(x)$, and $n^{-1}p(x) \leq p(n^{-1}x)$ *for all integer* $n > 0$,

6. $p(x - y) \leq p(x - z) + p(z - y)$ *for all* $z \in \theta^{-1}(x) \cup \theta^{-1}(y)$.

PROOF. Repeat the argument given in the proof of [8, Theorem 1.3], or use the canonical decomposition of X.

REMARK 2.4. Note that if p is a seminorm on X, then we also have $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in K$ and $x \in X$.

DEFINITION 2.5. *An ordered pair* $X(P) = (X, P)$, *where* X *is a generalized vector space over* K *and* P *is a nonvoid family of preseminorms (seminorms) on* X, *will be called a generalized preseminormed (seminormed) space over* K.

If X(P) is a generalized preseminormed space and

$$\Gamma \in \{\theta, E, \theta^{-1}\},$$

then the weakest topology $\mathcal{T}_P^{\Gamma}$ on X for which all the balls

$$B_p(x,r) = \{y \in \Gamma(x): p(x - y) < r\},$$

where $p \in P$, $x \in X$ and $r > 0$, are open will be called the canonical Γ- topology induced by P on X.

REMARK 2.6. If $X(P)$ is a preseminormed space, then by (ii) in Remark 1.9, the topology $\mathcal{T}_P^{\Gamma}$ is identical to the topology $\mathcal{T}_P$ given in [8, Definition 2.1].

THEOREM 2.7. *Let* $X(P)$ *be a generalized preseminormed space and* $x \in V \subset X$. *Then the following properties are equivalent:*

(i) V *is a neighborhood of* x *in* $X(\mathcal{T}_P^{\Gamma})$;

(ii) *there exist* $\{p_k\}_{k=1}^{n} \subset P$ *and* $r>0$ *such that*

$$\bigcap_{k=1}^{n} B_{p_k}^{\Gamma}(x,r) \subset V.$$

PROOF. Straightforward, but a lengthy computation with balls which needs (6) from Theorem 2.3 and the fact that Γ is a preordering.

REMARK 2.8. Note that if P is directed in the sense that for each $p_1, p_2 \in P$ there exists $p \in P$ such that $p_1 \prec p$ and $p_2 \prec p$, then (ii) can be simplified.

REMARK 2.9. By Definition 2.5 and Theorem 2.7, it is clear that

$$\mathcal{T}_P^{E} = \sup\{\mathcal{T}_P^{\Theta}, \mathcal{T}_P^{\Theta^{-1}}\},$$

namely, we have

$$B_p^E(x,r) = B_p^{\theta}(x,r) \cap B_p^{\theta^{-1}}(x,r)$$

for any $p \in P$, $x \in X$ and $r > 0$.

THEOREM 2.10. *Let* $X(P)$ *be a generalized preseminormed space,* (x_α) *a net in* X *and* $x \in X$. *Then the following properties are equivalent:*

(i) $x \in \lim_\alpha x_\alpha$ *in* $X(T_P^\Gamma)$;

(ii) $x \in \underline{\lim}_\alpha \Gamma^{-1}(x_\alpha)$ *and* $\lim_\alpha p(x_\alpha - x) = 0$ *for all* $p \in P$.

PROOF. This is again an immediate consequence of Definition 2.5 and Theorem 2.7. (Recall that

$$\underline{\lim}_\alpha \Gamma^{-1}(x_\alpha) = \bigcup_\alpha \bigcap_{\beta \geq \alpha} \Gamma^{-1}(x_\beta).)$$

THEOREM 2.11. *Let* $X(P)$ *be a generalized preseminormed space over* $\mathbb{K}$, *and consider* X *to be equipped with the topology* T_P^Γ. *Then*

(i) *the addition* $+$ *is a continuous mapping of* $X \times X$ *into* X;

(ii) *the scalar multiplication* $\cdot$ *is a continuous mapping of* $\mathbb{K} \times X$ *into* X.

PROOF. Repeat the argument given in the proof of [8, Theorem 2.6], and use that Γ is compatible with the above operations.

THEOREM 2.12. *Let* $X(P)$ *be a generalized preseminormed space and* $p \in P$. *Then*

(i) p *is upper semicontinuous for* T_P^{θ};

(ii) p *is lower semicontinuous for* $T_P^{\theta^{-1}}$

(iii) p *is continuous for* T_P^E.

PROOF. If $x \in \lim_\alpha x_\alpha$ in $X(\mathcal{T}_P^\theta)$, then by Theorem 2.10

$$p(x_\alpha) \leq p(x_\alpha - x) + p(x),$$

for all sufficiently large α, and $\lim_\alpha p(x_\alpha - x) = 0$. Hence

$$\overline{\lim_\alpha}\, p(x_\alpha) \leq p(x)$$

which is equivalent to (i).

Assertion (ii) can be proved quite similarly, and assertion (iii) follows immediately from the former ones.

THEOREM 2.13. *Let $X(P)$ be a generalized preseminormed space. Then the topology induced by $\mathcal{T}_P^\Gamma$ on $E(x)$ is identical to $\mathcal{T}_{P|E(x)}$ for every $x \in X$.*

PROOF. This can be checked easily by using Theorem 2.7 or 2.10. (For $Y \subset X$, we define $P|Y = \{p|Y: p \in P\}$.)

THEOREM 2.14. *Let $X(P)$ be a generalized preseminormed space, and consider $E(x)$ to be equipped with the topology $\mathcal{T}_{P|E(x)}$ for every $x \in X$. Then $\mathcal{T}_P^E$ is the strongest (weakest) topology on X for which the identity mapping of $E(x)$ into X is continuous (open) for every $x \in X$.*

PROOF. This is again quite obvious by Theorem 2.10. (For some relevant facts, see [2, VI, 8] and [7].)

COROLLARY 2.15. *Let $X(P)$ be a generalized preseminormed space, and consider X to be equipped with the topology $\mathcal{T}_P^E$. Then a function f from X into a topological space Y is continuous (open) if and only if its restriction to $E(x)$ is continuous (open) for every $x \in X$.*

DEFINITION 2.16. *A family P of preseminorms on X will be called separating if for each $x \in X$ with $x \neq 0x$ there exists*

$p \in P$ *such that* $p(x) \neq 0$.

In particular, a preseminorm (seminorm) p will be called a prenorm (norm) if the singleton {p} is separating.

THEOREM 2.17. *Let* $X(P)$ *be a generalized preseminormed space. Then the topology* T_P^E *is Hausdorff if and only if* P *is separating.*

PROOF. This follows at once from Theorem 2.14 and [8, Theorem 2.10].

EXAMPLE 2.18. Let $X(P)$ be a preseminormed (seminormed) space, and for each $p \in P$, define the function $\dot{p}$ on $\dot{X}$ by

$$\dot{p}(x + M) = \inf_{m \in M} p(x + m).$$

Then $\dot{X}(\dot{P})$, where $\dot{P} = \{\dot{p}: p \in P\}$, is a generalized preseminormed (seminormed) space.

By Remark 1.11, only the subadditivity of $\dot{p}$ needs a proof. If $x + M, y + N \in \dot{X}$, then we have

$$\dot{p}((x + M) + (y + N)) \leq p(x + y + m + n) \leq p(x + m) + + p(y + n)$$

for all $m \in M$ and $n \in N$, whence it is clear that

$$\dot{p}((x + M) + (y + N)) \leq \dot{p}(x + M) + \dot{p}(y + N).$$

REMARK 2.19. Note that if

$$x + M \in \lim_{\alpha} (x_\alpha + M_\alpha)$$

in $\dot{X}(T_{\dot{P}}^{\theta})$ $(X(T_{\dot{P}}^{\theta^{-1}}))$, and $M'_\alpha (M')$ is a linear subspace of X such that $M_\alpha \subset M'_\alpha \subset (M \subset M')$ for each α, then we also have

$$x + M \in \lim_{\alpha} (x_\alpha + M'_\alpha) \quad (x + M' \in \lim_{\alpha}(x_\alpha + M_\alpha))$$

in $\dot{X}(\mathcal{T}^{\theta}_{\mathcal{P}})$ $(\dot{X}(\mathcal{T}^{\theta^{-1}}_{\mathcal{P}}))$.

REMARK 2.20. It is also noteworthy that if $\mathcal{P}$ is directed, then we have

$$x + M \in \lim_{\alpha} (x_\alpha + M_\alpha)$$

in $\dot{X}(\mathcal{T}^{E}_{\mathcal{P}})$, if and only if, there exists α_0 such that

$$x + M \in \lim_{\alpha \geq \alpha_0} (x_\alpha + M_\alpha)$$

in X/M, with respect to the usual quotient topology.

For a proof, see [9, Theorem 8.2 and Corollaries 3.5 and 8.6].

REMARK 2.21. It is well-known that the quotient topology on X/M is Hausdorff if and only if M is a closed subspace of X. Hence, by Theorem 2.17, it is clear that if $\mathcal{P}$ is directed, then $\dot{\mathcal{P}}$ is separating if and only if each linear subspace of X is closed. Since this can happen only in some very particular cases, we shall also need the following modification of Example 2.18.

EXAMPLE 2.22. Let $X(\mathcal{P})$ be a preseminormed (seminormed) space,

$$\hat{X} = \{x + M \in \dot{X}:\ X \setminus M \in \mathcal{T}_{\mathcal{P}}\},$$

and consider $\hat{X}$ to be endowed with the modified addition defined by

$$(x + M) \mathbin{\hat{+}} (y + N) = (x + y) + \overline{M + N}$$

and the usual scalar multiplication given in Example 1.10. Then $\hat{X}(\hat{\mathcal{P}})$, where $\hat{\mathcal{P}} = \dot{\mathcal{P}}|\hat{X}$, is a generalized preseminormed (seminormed) space with zero and infinity such that

$$\Theta(x + M) = \{y + N \in \hat{X} : M \subset N\}$$

for any $x + M \in \hat{X}$. Moreover, $\hat{P}$ is separating if P is directed.

To check the first assertion, note that for any $A, B, C \subset \subset X$

$$\overline{\overline{A + B} + C} = \overline{A + B + C} = \overline{A + \overline{B + C}}$$

since T_P is translation-invariant. Moreover, that for closed linear subspaces M and N of X, $\overline{M + N} = N$ is equivalent to $M \subset N$. The second assertion is immediate from Remark 2.21.

3. CONTINUITY AND BOUNDEDNESS

DEFINITION 3.1. *A function* f *from one generalized preseminormed space* X(P) *into another* Y(Q) *will be called* Γ*-continuous (at* x*) if* f *is continuous (at* x*) for the topologies* T_P^Γ *and* T_Q^Γ.

THEOREM 3.2. *Let* f *be a linear function from one generalized preseminormed space* X(P) *into another* Y(Q) *and* $\Gamma = \Theta$ *or* E. *Then the following properties are equivalent:*

(i) f *is* Γ*-continuous;*

(ii) f *is* Γ*-continuous at* 0x *for every* $x \in X$;

(iii) $q \cdot f$ *is* Γ*-continuous at* 0x *for every* $x \in X$ *and* $q \in Q$.

PROOF. The implication (i) => (ii) is trivial. Moreover, by Theorems 2.12 and 2.3, it is clear that (ii) implies (iii).

To prove that (iii) also implies (i), suppose that (iii) holds and $x \in \lim_\alpha x_\alpha$ in $X(T_P^\Gamma)$. Then, by Theorem 2.11,

$$\lim_\alpha q(f(x_\alpha) - f(x)) = \lim_\alpha (q \cdot f)(x_\alpha - x) = (q \cdot f)(0x) = 0$$

for all $q \in Q$. Hence, by Theorem 2.10 and Remark 1.14, it is clear that $f(x) \in \lim_\alpha f(x_\alpha)$ in $X(\mathcal{T}_P^\Gamma)$. Consequently, (i) holds.

REMARK 3.3. Also by Theorem 2.10, it is clear that if f is Γ-continuous for $\Gamma = \Theta$ or Θ^{-1}, then f is also E-continuous.

THEOREM 3.4. *Let f be a linear function from one generalized preseminormed space $X(P)$ with zero into another $Y(Q)$. Then the following properties are equivalent:*

(i) *f is Θ-continuous at 0;*

(ii) *if* $\lim_\alpha p(x_\alpha) = 0$ *for all* $p \in P$, *then* $\lim_\alpha (q \circ f)(x_\alpha) = 0$ *for all* $q \in Q$;

(iii) *f is Γ-continuous for any* $\Gamma \in \{\Theta, E, \Theta^{-1}\}$.

PROOF. Observe that $\Theta(0) = X$, and use a similar argument as in the proof of Theorem 3.2.

REMARK 3.5. Note that implication (ii) => (iii) does not requires X to have a zero.

THEOREM 3.6. *Let f be a linear function from one generalized seminormed space $X(P)$ with zero into another $Y(Q)$. Then the following properties are equivalent:*

(i) *f is Θ-continuous;*

(ii) *for each* $q \in Q$, *there exist* $\{p_k\}_{k=1}^n \subset P$ *and* $M > 0$ *such that*

$$q \circ f \leq M \max_{1 \leq k \leq n} p_k .$$

PROOF. By Theorem 3.4, it is clear that (ii) implies (i). Suppose now that (i) holds and $q \in Q$. Then, by Theorem 3.2, $\rho = q \circ f$ is Θ-continuous at 0. Thus, by Theorem 2.7, there exist $\{p_k\}_{k=1}^n \subset P$ and $r > 0$ such that with the notation $p = \max_{1 \leq k \leq n} p_k$

$$B_p^{\Theta}(0,r) \subset \rho^{-1}(]-1,1[)$$

holds. Since $\Theta(0) = X$, this is equivalent to the statement that $p(x) < r$ implies $\rho(x) < 1$. Hence, using a standard argument on seminorms, it is easy to infer that $\rho \leq \frac{1}{r}\, p$.

DEFINITION 3.7. *Two nonvoid families $\mathcal{P}$ and $\mathcal{Q}$ of preseminorms on X will be called Γ-equivalent if $\mathcal{T}_{\mathcal{P}}^{\Gamma} = \mathcal{T}_{\mathcal{Q}}^{\Gamma}$.*

If $\mathcal{P}$ and $\mathcal{Q}$ are Γ-equivalent for any $\Gamma \in \{\Theta, E, \Theta^{-1}\}$, then we shall simply say that $\mathcal{P}$ and $\mathcal{Q}$ are equivalent.

REMARK 3.8. Using the particular case of the above results when $X = Y$ and f is the identity function of X, one can obtain several necessary or sufficient conditions for the Γ-equivalence of $\mathcal{P}$ and $\mathcal{Q}$ with various Γ.

For instance, Theorem 3.4 together with Remark 3.5 shows that $\mathcal{P}$ and $\mathcal{Q}$ are equivalent if for any net (x_α) in X, $\lim_\alpha p(x_\alpha) = 0$ for all $p \in \mathcal{P}$ if and only if $\lim_\alpha q(x_\alpha) = 0$ for all $q \in \mathcal{Q}$. This simple fact can be used to prove the following important

THEOREM 3.9. *Let $\mathcal{P}$ be a nonvoid countable (finite) family of preseminorms (seminorms) on X, then there exists a preseminorm (seminorm) p on X such that $\mathcal{P}$ and $\{p\}$ are equivalent.*

PROOF. If $\mathcal{P} = \{p_n\}_{n=1}^{\infty}$ (resp. $\mathcal{P} = \{p_k\}_{k=1}^{m}$), define

$$p = \sum_{n=1}^{\infty} \min\{p_n, 2^{-n}\} \quad (\text{resp. } p = \sum_{k=1}^{m} p_k).$$

To check that p has the required property, break the proof into a succession of steps according to [8, Theorem 3.6].

REMARK 3.10. Note that p is a prenorm (norm) if and only if $\mathcal{P}$ is separating.

DEFINITION 3.11. *A subset A of a generalized preseminor-*

med space X(P) *will be called bounded if* $A = \phi$ *or*

$$\lim_{\lambda \to 0} \sup_{x \in A} p(\lambda x) = 0$$

for all $p \in P$.

REMARK 3.12. Property (i) in Definition 2.1 implies that each finite subset of a generalized preseminormed space X(P) is bounded.

REMARK 3.13. If A is a bounded subset of a generalized preseminormed space X(P), then by (5) in Theorem 2.3, it is clear that $\sup p(A) < +\infty$ for all $p \in P$. Note that if X(P) is a seminormed space, then the converse is also true.

REMARK 3.14. It is also noteworthy that if A is a bounded subset of a generalized preseminormed space X(P) such that $nA \subset A$ for all integers $n > 0$ (or more specially $A + A \subset A$), then $p(x) = 0$, for all $x \in A$ and $p \in P$, and thus, in particular, $x = 0x$ for all $x \in A$, if P is separating.

THEOREM 3.15. *Let* X(P) *be a generalized preseminormed space over* K *with zero and* $A \subset X$. *Then the following properties are equivalent:*

(i) A *is a bounded subset of* X(P);

(ii) *for each neighborhood* V *of* 0 *in* $X(T_P^{\theta})$ *there exists an integer* $n > 0$ *such that* $A \subset nV$;

(iii) *if* (x_α) *is a net in* A *and* (λ_α) *is a net in* K *such that* $\lim_\alpha \lambda_\alpha = 0$, *then* $0 \in \lim_\alpha \lambda_\alpha x_\alpha$ *in* $X(T_P^{\theta})$.

PROOF. The implications (i) => (ii) and (ii) => (iii) are quite obvious by Theorem 2.7 and (2) in Theorem 2.3.

To prove that (iii) also implies (i), suppose indirectly that (iii) is true, but (i) is false. Then, there exists $\varepsilon > 0$ such that for each integer $n > 0$, there exist $|\lambda_n| < n^{-1}$ and

$x_n \in A$ such that $p(\lambda_n x_n) \geq \varepsilon$. Hence, by Theorem 2.10, it is clear that $0 \notin \lim_{n\to\infty} \lambda_n x_n$ in $X(T_P^\Theta)$ which contradicts (iii).

DEFINITION 3.16. *A linear function* f *from one generalized preseminormed space* X(P) *into another* Y(Q) *will be called bounded if* f(A) *is a bounded subset of* Y(Q) *for every bounded subset* A *of* X(P).

THEOREM 3.17. *Let* f *be a* Θ*-continuous linear function from one generalized preseminormed space* X(P) *with zero into another* Y(Q). *Then* f *is bounded.*

PROOF. Apply (iii) in Theorem 3.15. (Note that only the homogenity of f is essential.)

COROLLARY 3.18. *If* P *and* Q *are* Θ*-equivalent families of preseminorms on a generalized vector space* X *with zero. Then* X(P) *and* X(Q) *have the same bounded sets.*

To establish a certain converse to Theorem 3.17, we need the following

LEMMA 3.19. *Let* X(P) *be a generalized preseminormed space and* (x_n) *be a sequence in* X. *Then, for* $\Gamma = \Theta$ *or* E, *the following assertions hold:*

(i) *If* $\lim_{n\to\infty} x_n \neq \phi$ in $X(T_P^\Gamma)$, *then* $\{x_n\}_{n=1}^\infty$ *is a bounded subset of* X(P).

(ii) *If* $0x \in \lim_{n\to\infty} x_n$ in $X(T_P^\Gamma)$ *for some* $x \in X$ *and* P *is countable, then there exists an unbounded, nondecreasing sequence* (k_n) *of positive integers such that* $0x \in \lim_{n\to\infty} k_n x_n$ *in* $X(T_P^\Gamma)$.

PROOF. Suppose that $x \in \lim_{n\to\infty} x_n$ in $X(T_P^\Gamma)$, and let $p \in P$ and $\varepsilon>0$. Then, by Theorem 2.10, there exists n_0 such that $p(x_n - x) < \varepsilon 3^{-1}$ for all $n > n_0$. Moreover, by Remark 3.12, there exists $0 < \delta \leq 1$ such that $p(\lambda x_n - \lambda x) < \varepsilon 3^{-1}$ if $n \leq n_0$, and

$p(\lambda x) < \varepsilon 3^{-1}$ for all $|\lambda| < \delta$. Hence, we can infer that

$$\sup_n p(\lambda x_n) \leq \sup_{n \leq n_0} p(\lambda x_n - \lambda x) + \sup_{n > n_0} p(x_n - x) + p(\lambda x) < \varepsilon$$

for all $|\lambda| < \delta$. Thus $\lim_{\lambda \to 0} \sup_n p(\lambda x_n) = 0$.

Suppose now that $0x \in \lim_{n\to\infty} x_n$ in $X(\mathcal{T}_P^\Gamma)$ and P is countable. Then, by Theorem 3.9, we may suppose that $P = \{p\}$. Moreover, by Theorem 2.10, $\lim_{n\to\infty} p(x_n) = 0$. Thus, there exists a subsequence (x_{n_k}) of (x_n) such that $p(x_{n_k}) < k^{-2}$ if $n > n_k$. Put $k_n = 1$ if $n \leq n_1$, and $k_n = k$ if $n_k < n \leq n_{k+1}$. Then, for $n_k < n \leq n_{k+1}$,

$$p(k_n x_n) = p(kx_n) \leq kp(x_n) < k^{-1},$$

and thus $\lim_{n\to\infty} p(k_n x_n) = 0$. Hence, again by Theorem 2.10, it is clear that $0x \in \lim_{n\to\infty} k_n x_n$ in $X(\mathcal{T}_P^\Gamma)$.

THEOREM 3.20. *Let f be a bounded linear function from one generalized preseminormed space X(P) with zero into another Y(Q), and suppose that P is countable. Then f is Γ-continuous for any* $\Gamma \in \{\Theta, E, \Theta^{-1}\}$.

PROOF. By Theorem 3.4, we need show only that f is Θ-continuous at 0. Moreover, since $X(\mathcal{T}_P^\Theta)$ satisfies now the first axiom of countability by Theorem 2.7, we may use sequences instead of nets.

Suppose that (x_n) is a sequence in X such that $0 \in \lim_{n\to\infty} x_n$ in $X(\mathcal{T}_P^\Theta)$. Then, by (ii) in Lemma 3.19, there exists a null sequence (λ_n) of positive numbers such that $0 \in \lim_{n\to\infty} \lambda_n^{-1} x_n$ in $X(\mathcal{T}_P^\Theta)$. Moreover, by (i) in Lemma 3.19, $\{\lambda_n^{-1} x_n\}_{n=1}^\infty$ is a bounded subset of X(P). Thus, by the assumption, $\{f(\lambda_n^{-1} x_n)\}_{n=1}^\infty$ is a bounded subset of Y(Q). Hence, using (iii) in Theorem 3.15, we can infer that

$$f(0) \in \lim_{n\to\infty} \lambda_n f(\lambda_n^{-1} x_n) = \lim_{n\to\infty} f(x_n)$$

in $Y(\mathcal{T}_P^\Theta)$.

EXAMPLE 3.21. Let X be a generalized vector space and $Y = Y(Q)$ be a generalized preseminormed (seminormed) space. Let $\mathcal{A}$ be a nonvoid family of nonvoid subsets of X, and denote by $\mathcal{B}_{\mathcal{A}} = \mathcal{B}_{\mathcal{A}}(X,Y)$ the family of all functions $f \in \mathcal{L}(X,Y)$ for which $f(A)$ is a bounded subset of $Y(Q)$ for every $A \in \mathcal{A}$. Consider $\mathcal{B}_{\mathcal{A}}$ to be endowed with the operations given in Example 1.13, and for each $A \in \mathcal{A}$ and $q \in Q$, define the function q_A on $\mathcal{B}_{\mathcal{A}}$ by

$$q_A(f) = \sup_{x \in A} q(f(x)).$$

Then $\mathcal{B}_{\mathcal{A}}(Q_{\mathcal{A}})$, where

$$Q_{\mathcal{A}} = \{q_A : A \in \mathcal{A} , q \in Q\},$$

is a generalized preseminormed (seminormed) space.

Note that this is an immediate consequence, and also the main motivation, of Definition 3.11.

REMARK 3.22. If $X = X(\mathcal{P})$ is also a generalized preseminormed space and $\mathcal{A}$ is the family of all nonvoid bounded subsets of X, then $\mathcal{B}_{\mathcal{A}}(X,Y)$ is just the family of all bounded linear function from X into Y.

Thus, if in addition X has a zero and $\mathcal{P}$ is countable, then by Theorems 3.18 and 3.20, $\mathcal{B}_{\mathcal{A}}(X,Y)$ is identical to the family of all θ-continuous linear functions from X into Y.

REMARK 3.23. Note that if Q is directed, then $Q_{\mathcal{A}}$ is also directed.

Note also that if Q is separating and $\bigcup \mathcal{A} = X$, then $Q_{\mathcal{A}}$ is also separating.

REMARK 3.24. In connection with Example 3.21, it is also useful to note that if $\mathcal{B} \subset \mathcal{A}$ such that for each $A \in \mathcal{A}$, there exist $\{B_k\}_{k=1}^{n} \subset \mathcal{B}$ and a family $\{\Lambda_k\}_{k=1}^{n}$ of bounded subset of K such that

$$A \subset \sum_{k=1}^{n} \Lambda_k B_k ,$$

then $Q_{\mathcal{A}}$ and $Q_{\mathcal{B}} = \{q_B : B \in \mathcal{B},\ q \in Q\}$ are equivalent.

Thus, in particular, if $X = X(\mathcal{P})$ is a generalized seminormed space with zero such that $\mathcal{P} = \{p\}$, $\mathcal{A}$ is the family of all nonvoid bounded subsets of X and

$$B = \{x \in X : p(x) < 1\},$$

then $Q_{\mathcal{A}}$ and $Q_B = \{q_B : q \in Q\}$ are equivalent.

To check the above assertions use Remark 3.8 and Theorems 2.3 and 3.15.

4. COMPLETENESS

DEFINITION 4.1. *A net* (x_α) *in a generalized preseminormed space* $X(\mathcal{P})$ *will be called a* Γ*-Cauchy net if*

(i) $\underline{\lim}_{\alpha}\ \Gamma^{-1}(x_\alpha) \neq \phi$;

(ii) $\lim_{(\alpha,\beta)} p(x_\alpha - x_\beta) = 0$ *for all* $p \in \mathcal{P}$.

REMARK 4.2. Note that if X has a zero (infinity) and $\Gamma = \Theta(\Theta^{-1})$, or X is a vector space, then condition (i) is automatically satisfied.

THEOREM 4.3. *Let* (x_α) *be a net in a generalized preseminormed space* $X(\mathcal{P})$ *and* $\Gamma = \Theta$ *or* E. *Then the following properties are equivalent:*

(i) (x_α) *is a* Γ*-Cauchy net in* $X(\mathcal{P})$;

(ii) $0x \in \lim_{(\alpha,\beta)}(x_\alpha - x_\beta)$ *in* $X(\mathcal{T}_{\mathcal{P}}^{\Gamma})$ *for some* $x \in X$.

PROOF. This is quite obvious by Theorem 2.10. (Note that the implication (i)$\Rightarrow$ (ii) is also true for $\Gamma = \Theta^{-1}$.)

COROLLARY 4.4. *Let* f *be a* Γ*-continuous linear function from one generalized preseminormed space* $X(\mathcal{P})$ *into another*

$Y(Q)$ and (x_α) be a Γ-Cauchy net in $X(P)$ with $\Gamma = \Theta$ or E. Then $(f(x_\alpha))$ is a Γ-Cauchy net in $Y(Q)$.

PROOF. By Theorem 4.3, $0x \in \lim_{(\alpha,\beta)} (x_\alpha - x_\beta)$ in $X(T_P^\Gamma)$ for some $x \in X$. Hence, since f is Γ-continuous, it follows that

$$0f(x) = f(0x) \in \lim_{(\alpha,\beta)} f(x_\alpha - x_\beta) = \lim_{(\alpha,\beta)} (f(x_\alpha) - f(x_\beta))$$

in $Y(T_Q^\Gamma)$. Thus, again by Theorem 3.4, $(f(x_\alpha))$ is a Γ-Cauchy net in $Y(Q)$.

COROLLARY 4.5. *Let P and Q be Γ-equivalent families of preseminorms on X with $\Gamma = \Theta$ or E. Then $X(P)$ and $X(Q)$ have the same Γ-Cauchy nets.*

COROLLARY 4.6. *Let (x_α) be a net in a generalized preseminormed space $X(P)$ such that $\lim_\alpha x_\alpha \neq \phi$ i $X(T_P^\Gamma)$, where $\Gamma = \Theta$ or E. Then (x_α) is a Γ-Cauchy net in $X(P)$.*

PROOF. If $x \in \lim_\alpha x_\alpha$ in $X(T_P^\Gamma)$, then we also have $x \in \lim_{(\alpha,\beta)} x_\alpha$ and $x \in \lim_{(\alpha,\beta)} x_\beta$, and hence by Theorem 2.11 $0x = x - x \in \lim_{(\alpha,\beta)} (x_\alpha - x_\beta)$ in $X(T_P^\Gamma)$. Thus, by Theorem 4.3, (x_α) is a Γ-Cauchy net in $X(P)$.

THEOREM 4.7. *Let (x_α) be a Γ- Cauchy net in a generalized preseminormed space $X(P)$ and $x \in \underline{\lim}_\alpha \Gamma^{-1}(x_\alpha)$ such that x is a cluster point of (x_α) in $X(T_P^{\Theta^{-1}})$. Then $x \in \lim x_\alpha$ in $X(T_P^\Gamma)$.*

PROOF. A standard argument and again Theorem 2.10 can be applied.

DEFINITION 4.8. *A subset A of a generalized preseminormmed space $X(P)$ will be called Γ-complete if $\lim_\alpha x_\alpha \cap A \neq \phi$ for any net (x_α) in A which is a Γ-Cauchy net in $X(P)$.*

In particular, the generalized preseminormed space $X(\mathcal{P})$ will be called Γ-complete if X is a Γ-complete subset of $X(\mathcal{P})$.

THEOREM 4.9. *Let $X(\mathcal{P})$ be a generalized preseminormed space. Then the following assertions hold:*

(i) If A is a closed subset of $X(T_{\mathcal{P}}^{\Gamma})$ and $X(\mathcal{P})$ is Γ-complete, then A is a Γ-complete subset of $X(\mathcal{P})$.

(ii) If A is a Γ-complete subset of $X(\mathcal{P})$ with $\Gamma = \Theta$ or E, and $T_{\mathcal{P}}^{\Gamma}$ is Hausdorff, then A is a closed subset of $X(T_{\mathcal{P}}^{\Gamma})$.

PROOF. Left to the reader.

THEOREM 4.10. *Let $X(\mathcal{P})$ and $Y(\mathcal{Q})$ be generalized preseminormed spaces and $\Gamma = \Theta$ or E. Suppose that there exists a linear injection f from X onto Y such that both f and f^{-1} are Γ-continuous. Then $X(\mathcal{P})$ is Γ-complete if and only if $Y(\mathcal{Q})$ is Γ-complete.*

PROOF. This is quite obvious by Corollary 4.4.

COROLLARY 4.11. *Let $\mathcal{P}$ and $\mathcal{Q}$ be Γ-equivalent families of preseminorms on X with $\Gamma = \Theta$ or E. Then $X(\mathcal{P})$ is Γ-complete if and only if $X(\mathcal{Q})$ is Γ-complete.*

THEOREM 4.12. *Let $X(\mathcal{P})$ be a a generalized preseminormed space with infinity. Then $X(\mathcal{P})$ is Θ^{-1}-complete.*

PROOF. For any net (x_α) in X, we have $x_\alpha \in \Theta^{-1}(\infty)$ and

$$p(x_\alpha - \infty) = p(\infty) = p(0\infty) = 0$$

for all α and $p \in \mathcal{P}$, whence by Theorem 2.10, $\infty \in \lim_\alpha x_\alpha$ in $X(T_{\mathcal{P}}^{\Theta^{-1}})$.

COROLLARY 4.13. *Let* $X(P)$ *be a preseminormed space. Then the generalized preseminormed spaces* $\dot{X}(\dot{P})$ *and* $\hat{X}(\hat{P})$ *are* Θ^{-1}*- complete.*

COROLLARY 4.14. *Let* X *be a generalized vector space,* $Y = Y(Q)$ *a generalized preseminormed space with infinity and* A *a nonvoid family of nonvoid subsets of* X*. Then the generalized preseminormed space* $B_A(X,Y)(Q_A)$ *is* Θ^{-1}*-complete.*

PROOF. Note that the function taking the value ∞ at every $x \in X$ belongs to $B_A(X,Y)$.

COROLLARY 4.15. *Let* X *be a generalized vector space,* $Y = Y(Q)$ *a preseminormed space, and* A *a nonvoid family of nonvoid subsets of* X*. Then the generalized preseminormed spaces* $B_A(X,\dot{Y})(\dot{Q}_A)$ *and* $B_A(X,\hat{Y})(\hat{Q}_A)$ *are* Θ^{-1}*-complete.*

THEOREM 4.16. *Let* $X(P)$ *be a generalized preseminormed space. Then* $X(P)$ *is* E*-complete if and only if the preseminormed space* $E(x)(P|E(x))$ *is complete for every* $x \in X$*.*

PROOF. This is immediate consequence of Definitions 4.1 and 4.8 and Theorem 2.10.

COROLLARY 4.17. *Let* $X(P)$ *be a complete preseminormed space, such that* P *is countable and directed. Then the generalized preseminormed spaces* $\dot{X}(\dot{P})$ *and* $X(P)$ *are* E*-complete.*

PROOF. In this case, all the quotient spaces of X are complete [3, Lemma 11.3], and thus Theorem 4.16 can be applied. (Recall that $E(x + M) = X/M$ for any $x + M \in \dot{X}$ or $\hat{X}$.)

REMARK 4.18. In a continuation of [9], we shall prove that if $X(P)$ is as in Corollary 4.17 and f is a linear function from $\dot{X}$ onto Y, then

$$Y(Q) = \lim \operatorname{proj}_{f^{-1}} X(P)$$

is also complete. This preseminormed analogue of [3, Lemma 1.13] can be used to derive Corollary 4.17 more directly.

THEOREM 4.19. *Let* X *be a vector space,* $Y = Y(Q)$ *a* Γ*-complete generalized preseminormed space with* $\Gamma = E$ *or* Θ^{-1}*, and* $\mathcal{A}$ *a family of nonvoid subsets of* X *such that* $X = \bigcup \mathcal{A}$. *Then the generalized preseminormed space* $\mathcal{B}_{\mathcal{A}}(X,Y)(Q_{\mathcal{A}})$ *is also* Γ*-complete.*

PROOF. *Let* (f_α) be a Γ-Cauchy net in $\mathcal{B}_{\mathcal{A}}(Q_{\mathcal{A}})$, where $\mathcal{B}_{\mathcal{A}} = \mathcal{B}_{\mathcal{A}}(X,Y)$. Then, since $\mathcal{A}$ covers X, $(f_\alpha(x))$ is a Γ-Cauchy net in $Y(Q)$ for each $x \in X$. Thus, since $Y(Q)$ is Γ-complete, we may define a relation F from X into Y such that

$$F(x) = \lim_\alpha f_\alpha(x)$$

in $Y(\mathcal{T}_Q^\Gamma)$ for all $x \in X$. Using Theorem 2.11, it is easy to see that F is linear in the sense that

$$F(x) + F(y) \subset F(x+y) \quad \text{and} \quad \lambda F(x) \subset F(\lambda x)$$

for all $x,y \in X$ and $\lambda \in \mathbb{K}$. Moreover, repeating the argument used in the proof of [6, Theorem 4.1], we can state that F has a linear selection f.

In the sequel, we shall show that $f \in \lim_\alpha f_\alpha$ in $\mathcal{B}_{\mathcal{A}}(\mathcal{T}^\Gamma_{Q_{\mathcal{A}}})$. For this, suppose that $q \in Q$, $A \in \mathcal{A}$ and $\varepsilon > 0$. Since $f(0) \in \lim_\alpha f_\alpha(0)$ in $Y(\mathcal{T}_Q^\Gamma)$, there exists α_0 such that $f_\alpha(0) \in \Gamma(f(0))$ for all $\alpha \geq \alpha_0$. Hence, by Remark 1.14, it follows that

$$f_\alpha(x) \in \Gamma(f(x))$$

for all $\alpha \geq \alpha_0$ and $x \in X$. Moreover, since (f_α) is a Γ-Cauchy net in $\mathcal{B}_{\mathcal{A}}(Q_{\mathcal{A}})$, there exists $\alpha_1 \geq \alpha_0$ such that $q(f_\alpha(x) - f_\beta(x)) < 3^{-1}\varepsilon$ for all $\alpha, \beta \geq \alpha_1$ and $x \in A$. Hence, using Theorem 2.12, we can infer that

$$q(f_\alpha(x) - f(x)) \leq \lim_\beta q(f_\alpha(x) - f_\beta(x)) \leq 3^{-1}\varepsilon$$

for all $\alpha \geq \alpha_1$ and $x \in A$. On the other hand, since $f_{\alpha_1}(A)$ is a bounded subset of $Y(Q)$, there exists $0 \leq \delta < 1$ such that

$$q(\lambda f_{\alpha_1}(x)) < 2^{-1}\varepsilon$$

for all $|\lambda| < \delta$ and $x \in A$. Thus, we have

$$q(\lambda f(x)) \leq q(f_{\alpha_1}(x) - f(x)) + q(\lambda f_{\alpha_1}(x)) < \varepsilon$$

for all $|\lambda| < \delta$ and $x \in A$. Hence, it is clear that $f \in B_A$. Moreover, from the above results by Theorem 2.10, it follows, at once, that we also have $f \in \lim_\alpha f_\alpha$ in $B_A(T^\Gamma_{Q_A})$.

REMARK 4.20. The same assertion can be proved if X is a generalized vector space, but T^Γ_Q is Hausdorff.

COROLLARY 4.21. *Let X be a vector space,* $Y = Y(Q)$ *a complete preseminormed space such that* Q *is countable and directed, and* $\mathcal{A}$ *a family of nonvoid subsets of* X *such that* $\dot{X} = \bigcup \dot{\mathcal{A}}$ *Then the generalized preseminormed spaces* $B_A(X,Y)(\dot{Q}_A)$ *and* $B_A(X,\hat{Y})(\hat{Q}_A)$ *are E-complete.*

PROOF. In this case, by Corollary 4.17, $\dot{Y}(\dot{Q})$ and $\hat{Y}(\hat{Q})$ are E-complete, and thus Theorem 4.19 can be applied.

5. APPLICATIONS TO LINEAR RELATIONS

THEOREM 5.1. *Let* S *be a linear relation from one preseminormed space* X(P) *into another* Y(Q) *with* Q *directed. Then the following properties are equivalent:*

(i) S *is lower semicontinuous* [5];

(ii) φ_S *is Θ-continuous as a function of* X(P) *into* $\dot{Y}(\dot{Q})$

PROOF. This follows immediately from [9, Theorem 2.1] and Theorem 3.2, since we have

$$q * S = \dot{q} \cdot \varphi_S$$

for any $q \in Q$.

REMARK 5.2. Note that implication (i) => (ii) does not require Q to be directed.

Note also that if (ii) holds, then by Theorem 3.4, φ_S is Γ-continuous for any $\Gamma \in \{\Theta, E, \Theta^{-1}\}$.

DEFINITION 5.3. *A linear relation* S *from one preseminormed space* X(P) *into another* Y(Q) *will be called bounded if*

$$\lim_{\lambda \to 0} \sup_{x \in A} (q*S)(\lambda x) = 0$$

for every nonvoid bounded subset A *of* X(P) *and* $q \in Q$.

REMARK 5.4. Note that if a linear relation from one preseminormed space into another maps bounded sets into bounded sets, then it is necessarily a function.

This fact and the forthcoming assertions explain the above definition.

THEOREM 5.5. *Let* S *be a linear relation from one preseminormed space* X(P) *into another* Y(Q). *Then the following properties are equivalent:*

(i) S *is bounded;*

(ii) φ_S *is bounded as a function of* X(P) *into* $\dot{Y}(\dot{Q})$.

PROOF. This is an immediate consequence of the corresponding definitions.

THEOREM 5.6. *Let* S *be a linear relation from one pre-*

seminormed space $X(\mathcal{P})$ *into another* $Y(Q)$. *Then the following assertions hold:*

(i) *If* S *is lower semicontinuous, then* S *is bounded.*

(ii) *If* S *is bounded,* $\mathcal{P}$ *is countable and* Q *is directed, then* S *is lower semicontinuous.*

PROOF. This can be derived at once from Theorems 3.17 and 3.20 by using Theorems 5.1 and 5.5 and also Remark 5.2.

THEOREM 5.7. *Let* X *be a vector space and* $Y = Y(Q)$ *be a preseminormed (seminormed) space. Let* $\mathcal{A}$ *be a nonvoid family of nonvoid subsets of* X, *and denote* $\mathcal{B}_{\mathcal{A}} = \mathcal{B}_{\mathcal{A}}(X,Y)$ *the family of all relations* $S \in L(X,Y)$ *for which*

$$\lim_{\lambda \to 0} \sup_{x \in A} (q*S)(\lambda x) = 0$$

for every $A \in \mathcal{A}$ *and* $q \in Q$. *Consider* $\mathcal{B}_{\mathcal{A}}$ *to be endowed with the operations given in Example 1.12, and for each* $A \in \mathcal{A}$ *and* $q \in Q$, *define*

$$q_A(S) = \sup_{x \in A} (q * S)(x).$$

Then $\mathcal{B}_{\mathcal{A}}(Q_{\mathcal{A}})$, *where*

$$Q_{\mathcal{A}} = \{q_A : A \in \mathcal{A}, q \in Q\}$$

is a generalized preseminormed (seminormed) space.

PROOF. Note that $S \in \mathcal{B}_{\mathcal{A}}$ if and only if $\varphi_S \in \dot{\mathcal{B}}_{\mathcal{A}}(X,\dot{Y})$, and moreover

$$q_A(S) = \dot{q}_A(\varphi_S)$$

for any $A \in \mathcal{A}$, $q \in Q$ and $S \in \mathcal{B}_{\mathcal{A}}$.

REMARK 5.8. Note that if $X = X(\mathcal{P})$ is also a preseminormed space and $\mathcal{A}$ is the family of all nonvoid bounded subsets of X, then $\mathcal{B}_{\mathcal{A}}(X,Y)$ is just the family of all bounded linear relations from X into Y.

Thus, if in addition $\mathcal{P}$ is countable and Q is directed, then by Theorem 5.6, $\mathcal{B}_{\mathcal{A}}(X,Y)$ is identical to the family of all lower semicontinuous linear relations from X into Y.

THEOREM 5.9. *Let X be a vector space and Y = Y(Q) be a preseminormed (seminormed) space. Let $\mathcal{A}$ be a nonvoid family of nonvoid subsets of X, and denote by $\hat{\mathcal{B}}_{\mathcal{A}} = \hat{\mathcal{B}}_{\mathcal{A}}(X,Y)$ the family of all relations $S \in \mathcal{B}_{\mathcal{A}}(X,Y)$ which are closed-valued in the sense that S(x) is closed in Y for every $x \in X$. Consider $\hat{\mathcal{B}}_{\mathcal{A}}$ to be endowed with the modified addition defined by*

$$(S \hat{+} T)(x) = \overline{S(x) + T(x)}$$

and the usual scalar multiplication given in Example 1.12. Then $\hat{\mathcal{B}}_{\mathcal{A}}(\hat{Q}_{\mathcal{A}})$, where $\hat{Q}_{\mathcal{A}} = Q_{\mathcal{A}} \mid \hat{\mathcal{B}}_{\mathcal{A}}$, is a generalized preseminormed (seminormed) space with zero and infinity such that

$$\Theta(S) = \{T \in \hat{\mathcal{B}}_{\mathcal{A}} : S(0) \subset T(0)\}$$

for every $S \in \hat{\mathcal{B}}_{\mathcal{A}}$. Moreover, $\hat{Q}_{\mathcal{A}}$ is separating if Q is directed and $\bigcup \mathcal{A} = X$.

PROOF. Note that the mapping

$$S \mapsto \varphi_S \qquad (S \in \hat{\mathcal{B}}_{\mathcal{A}})$$

is a linear injection of $\hat{\mathcal{B}}_{\mathcal{A}}$ onto $B_{\mathcal{A}}(X,Y)$ which also preserves the corresponding preseminorms.

THEOREM 5.10. *Let X be a vector space, Y = Y(Q) a preseminormed space, and $\mathcal{A}$ a nonvoid family of nonvoid subsets of X. Then the generalized preseminormed spaces $\mathcal{B}_{\mathcal{A}}(X,Y)(Q_{\mathcal{A}})$ and $\hat{\mathcal{B}}_{\mathcal{A}}(X,Y)(\hat{Q}_{\mathcal{A}})$ are Θ^{-1}-complete.*

PROOF. This is an immediate consequence of Corollary 4.15.

THEOREM 5.11. *Let* X *be a vector space* Y = Y(Q) *a complete preseminormed space such that* Q *is countable and directed, and* A *a family of nonvoid subsets of* X *such that* X = $\bigcup$A. *Then the generalized preseminormed spaces* $B_A(X,Y)(Q_A)$ *and* $\hat{B}_A(X,Y)(\hat{Q}_A)$ *are* E-*complete.*

PROOF. This follows immediately from Corollary 4.21.

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REZIME

UOPŠTENI PRESEMINORMIRANI PROSTORI LINEARNIH MNOGOSTRUKOSTI I OGRANIČENE LINEARNE RELACIJE

U radu se uvodi uopšteni vektorski prostor sa familijom specijalnih funkcionela - preseminormi. Uvedeno uopštenje omogućava da se prenesu standardne teoreme o kompletnosti raznih prostora linearnih funkcija na prostore linearnih relacija. Tako se dokazuje i sledeće tvrdjenje:

Ako je $\mathcal{B}$ familija svih linearnih relacija od dole poluneprekidnih i zatvorenih vrednosti iz normiranog prostora X u Banachov prostor Y, tada je i $\hat{\mathcal{B}}$ Banachov prostor, gde su operacije definisane sa $S,T \in \hat{\mathcal{B}}$

$$(S \hat{+} T)(x) := \overline{S(x) + T(x)},$$

$$(\lambda S)(x) := S(\lambda x)$$

i normom

$$\|S\| := \sup_{\|x\| \leq 1} \inf_{y \in S(x)} \|y\| .$$