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SOME GENERALIZATIONS OF BROWDER'S FIXED POINT THEOREM IN TOPOLOGICAL SPACE

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ABSTRACT

In this note, using the method from [4], we prove a generalization of the well known Browder fixed point theorem for multivalued mappings in topological spaces. Similar results in topological vector spaces are proved by Tarafdar in [6]. A generalization of a coincidence point theorem from [5] is obtained.

H. Komiya introduced, in [4], the notion of convexity in an arbitrary topological space and obtained a generalization of the well known Browder's fixed point theorem for multivalued mappings, proved in [1].

We shall give some definitions and results from [4].

Let X be a Hausdorff topological space, $A(X)$ the family of all subsets of X and $F(X)$ the family of all finite subsets of X.

DEFINITION 1 [4]. *An H-operator on* X *is a mapping* $< >: A(X) \to A(X)$ *satisfying the following conditions:*

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(a) $<\emptyset> = \emptyset$.

(b) $<\{x\}> = \{x\}$, *for every* $x \in X$.

(c) $<<A>> = <A>$, *for every* $A \in A(X)$.

(d) $<A> = \bigcup\{<F> | F \subseteq A,\ F \in F(X)\}$.

A set $A \in A(X)$ is *convex* if $<A> = A$ and for $A \in A(X)$, the image $<A>$ of A is said to be the *convex hull* of A.

In [4], the following proposition is proved.

PROPOSITION.

(i) *An H-operator is monotone* $(A \subseteq B \Rightarrow <A> \subseteq <B>)$.

(ii) *The convex hull* $<A>$ *of* $A \in A(X)$ *is the smallest convex set containing* A.

(iii) $<X> = X$.

(iv) *If* $<C_i> = C_i$ $(i \in I)$ *then* $<\bigcap_{i\in I} C_i> = \bigcap_{i\in I} C_i$.

(v) *If* $<C_i> = C_i$, $i \in I$ *and for every* $i_1, i_2 \in I$ *there exists* $i \in I$ *with* $C_i \subseteq C_{i_1} \cap C_{i_2}$ *then* $<\bigcup_{i\in I} C_i> = \bigcup_{i\in I} C_i$.

Let R be the set of all functions $f : N \to \mathbb{R}$ (N is a countably infinite set) such that $\text{card}\{x | x \in N,\ f(x) \neq 0\} < \infty$. This implies that $R = \sum_{i\in N} R_i$, where $R_i = \mathbb{R}$. The topology and the linear structure on R are the usual ones.

Suppose that an H-operator $< >$ on a topological space X is given, and let $H(X) = \{<F> | F \in F(X)\}$.

DEFINITION 2 [4]. *For* $H \in H(X)$, *a mapping* $\phi : H \to R$ *is called a structure mapping on* H, *if it has the following properties:*

(a) *The mapping* ϕ *is an into-homeomorphism.*

(b) *If* $A \subseteq H$ *then* $\phi(<A>) = <\phi(A)>$, *where* $<\phi(A)>$ *is the usual convex hull of* $\phi(A)$ *in* R.

By S_H the set of all structures mappings on H is denoted. If $S_H \neq \emptyset$, for every $H \in \mathcal{H}(X)$ then $\Phi \in \prod_{H \in \mathcal{H}(X)} S_H$ is said to be a *structure* on X with respect to the H-operator $< >$.

DEFINITION 3 [4]. *A convex space* $(X,< >,\Phi)$ *is a triple consisting of a topological space* X, *an H-operator* $< >$ *on* X *and structure* Φ *on* X *with respect to* $< >$.

If $(X,< >,\Phi)$ is a convex space, $Y \subseteq X$ and Y is convex then $(Y,< >_Y,\Phi_Y)$ is a subspace of $(X,< >,\Phi)$ where the topology of Y is the induced topology, for $A \in \mathcal{A}(Y)$: $<A>_Y = <A>$ and $\Phi_Y = \Phi|\mathcal{H}(Y)$.

DEFINITION 4. *Let* X *be a topological space,* $< >$ *an H-operator on* X, $K \subseteq X$ *and* $A \subseteq X$. *We say that the set* A *is H-open on* K *if* $<G> \cap A$ *is open in* $<G>$ *for every* $G \in \mathcal{F}(K)$.

In [4] the following theorem is proved.

THEOREM 1. *Let* $(X,< >,\Phi)$ *be a compact convex space and* T *be a mapping from* X *into* $\mathcal{A}(X)\setminus\emptyset$, *where* $<Tx> = Tx$, *for every* $x \in X$. *If, for every* $y \in X$, $T^{-1}y = \{x|x \in X, y \in Tx\}$ *is open in* X, *then there exists* $x_o \in X$ *such that* $x_o \in Tx_o$.

LEMMA. *Let* $(X,< >,\Phi)$ *be a convex space,* K *a nonempty convex subset of* X *and* $F : K \to \mathcal{A}(X)$ *so that the following conditions are satisfied:*

(a) $x \in F(x)$, *for every* $x \in K$ *and* $C(F(x))$ *(the complement of* $F(x)$ *in* K*) is H-open on* K, *for every* $x \in K$.

(b) *For some* $x_o \in K$, $F(x_o)$ *is compact and* $F(x) \cap F(x_o)$ *is closed, for every* $x \in K$.

(c) *For every* $x \in K$ *the set:*

$$A(x) = \{y|y \in K, x \notin F(y)\}$$

is convex.

If, for every $G \in \mathcal{F}(K)$, $<G>$ *is compact then* $\bigcap_{x \in K} F(x) \neq \emptyset$.

PROOF: As in [6] we shall prove that $\bigcap_{i=1}^{n} F(x_i) \neq \emptyset$, for every finite subset $\{x_1, x_2, \ldots, x_n\}$ of K. Let us show that $\bigcap_{i=1}^{n} F(x_i) = \emptyset$ leads to a contradiction. Let $S = \langle x_1, x_2, \ldots, x_n \rangle$ and for every $x \in S$:

$$(1) \qquad B(x) = \{y \mid y \in S,\ x \notin F(y)\}.$$

Since $\bigcap_{i=1}^{n} F(x_i) = \emptyset$, it follows that for every $x \in S$ there exists $i_o \in \{1,2,\ldots,n\}$ so that $x \notin F(x_{i_o})$. From (1) it follows that $x_{i_o} \in B(x)$ and so $B(x) \neq \emptyset$, for every $x \in S$. Since $\langle A(x) \rangle = A(x)$ for every $x \in K$ and $B(x) = S \cap A(x)$, for every $x \in S$, using (iv) in the Proposition and (c) in Definition 1, we conclude that $\langle B(x) \rangle = B(x)$. Further let $T : S \to A(S)$ be defined in the following way: $T(x) = B(x)$, $x \in S$. Then:

$$T^{-1}(x) = \{y \mid y \in S,\ x \in T(y)\} = \{y \mid y \in S,\ x \in B(y)\} =$$

$$= \{y \mid y \in S,\ y \notin F(x)\} = S \cap C(F(x))$$

and from (a) it follows that $T^{-1}(x)$ is open in S. Applying the Theorem we conclude that there exists $x_o \in S$ so that $x_o \in Tx_o$ and so $x_o \notin F(x_o)$, which contradicts (a).

REMARK: The mapping F in the Lemma has the following property:

For every finite subset $\{x_1, x_2, \ldots, x_n\}$ of K:

$$\langle \{x_1, x_2, \ldots, x_n\} \rangle \subseteq \bigcup_{i=1}^{n} F(x_i).$$

If, on the contrary, for some finite subset $\{x_1, x_2, \ldots, x_n\} \subseteq K$

$$\langle \{x_1, x_2, \ldots, x_n\} \rangle \not\subseteq \bigcup_{i=1}^{n} F(x_i)$$

there exists $x \in \langle\{x_1,x_2,\ldots,x_n\}\rangle$ such that $x \notin F(x_i)$, for every $i \in \{1,2,\ldots,n\}$, which implies that $x_i \in A(x)$, for every $i \in \{1,2,\ldots,n\}$. Since $\langle A(x)\rangle = A(x)$, for every $x \in K$ and $x \in \langle\{x_1,x_2,\ldots,x_n\}\rangle$ it follows from $\langle\{x_1,x_2,\ldots,x_n\}\rangle \subseteq \subseteq A(x)$ ((i) in the Proposition) that $x \in A(x)$, which means that $x \notin F(x)$ and this contradicts (a) in the Lemma. Thus the mapping F from the Lemma is the so-called KKM mapping where co is replaced by $\langle\ \rangle$.

The following theorem is a generalizations of Tarafdar's result from [6].

THEOREM 2. *Let* $(X,\langle\ \rangle,\Phi)$ *be a convex space,* K *be a nonempty convex subset of* X*,* $T : K \to A(K)\setminus\emptyset$ *such that the following conditions are satisfied:*

(i) *For each* $x \in K$: $\langle T(x)\rangle = T(x)$.

(ii) *For some* $x_o \in K$, $C(T^{-1}(x_o))$ *is compact (the complement in* K*) and for each* $x \in K$ *the set* $C(T^{-1}(X)) \cap \cap C(T^{-1}(x_o))$ *is closed.*

(iii) *For every* $x \in K$ *the set* $T^{-1}(x)$ *is H-open on* K.

If, for every $G \in F(K)$, $\langle G\rangle$ *is compact, then there exists* $x_o \in K$ *such that* $x_o \in T(x_o)$.

PROOF: As in [6], let us define the mapping $F : K \to \to A(X)$ in the following way:

$$F(x) = C(T^{-1}(x)) = K\setminus T^{-1}(x), \text{ for every } x \in K.$$

If $x \notin T(x)$, for every $x \in K$ then $x \notin T^{-1}(x)$ and so $x \in F(x)$ for every $x \in K$. The mapping F satisfies all the conditions of the Lemma, which can be easily verified as in [6], and so there exists $u \in \bigcap_{x\in K} F(x)$. Then we have that $u \notin T^{-1}(x)$, for every $x \in K$, which contradicts $K = \bigcup_{x\in K} T^{-1}(x)$. From this we conclude that there exists $x_o \in K$ such that $x_o \in T(x_o)$.

Similarly as in [6], we can prove the following theorem using the Lemma.

THEOREM 3. *Let* $(X, \langle\ \rangle, \Phi)$ *be a convex space,* K *be a nonempty convex subset of* X *and* $T : K \to A(K)\setminus\emptyset$ *so that the following conditions are satisfied:*

(i) *For some* $x_o \in K$, $C(T(x_o))$ *is compact in* K *and for every* $x \in K$ *the set* $C(T(x)) \cap C(T(x_o))$ *is closed.*

(ii) *For every* $x \in K$, $\langle T^{-1}(x)\rangle = T^{-1}(x)$.

(iii) *For every* $x \in K$, $T(x)$ *is H-open on* X *and* $K = T(K)$.

If for every $G \in F(K)$, $\langle G\rangle$ *is compact then there exists* $x_o \in K$ *such that* $x_o \in T(x_o)$.

The following theorem is a generalization of the Theorem from [5].

THEOREM 4. *Let* $(X, \langle\ \rangle, \Phi)$ *be a convex space,* K *a nonempty convex subset of* X, F *a topological space and* $T, S : K \to A(F)$ *so that the following conditions are satisfied:*

1. T *is lower semicontinuous and* $\overline{T(K)}$ *is a compact subset of* F.

2. Sx *is open, for each* $x \in K$, $S^{-1}y \neq \emptyset$, *for every* $y \in \overline{T(K)}$ *and* $\langle S^{-1}Tx\rangle = S^{-1}Tx$, *for each* $x \in K$.

Then $T(u) \cap S(u) \neq \emptyset$, *for some* $u \in K$.

PROOF: The method of the proof is similar to that in [4]. Since $S^{-1}y \neq \emptyset$, for every $y \in \overline{T(K)}$, it follows that $\overline{T(K)} \subseteq \bigcup_{x\in K} \{Sx\}$ and since $\overline{T(K)}$ is compact and Sx is open, for every $x \in K$ it follows that:

$$\overline{T(K)} \subseteq \bigcup_{x\in F} Sx, \text{ for some } F \in F(K)$$

and so:

$$K \subseteq \bigcup_{i=1}^{n} T^{-1}(Sx_i), \text{ for some subset } \{x_1, x_2, \ldots, x_n\} \subseteq K.$$

Let $H = \{x_1, x_2, \ldots, x_n\}$ and $A_i = H \cap T^{-1}(Sx_i)$, $i \in \{1,2,\ldots,n\}$. Then $\{A_i\}_{i=1}^{n}$ is an open cover of H. Let $\tilde{H} = \phi(H) = \langle \phi(\{x_1, x_2, \ldots, x_n\})\rangle$, where $\phi = \Phi(H)$ is the structure mapping on H. If $\tilde{A}_i = \phi(A_i)$ $(i \in \{1,2,\ldots,n\})$ then $\{\tilde{A}_i\}_{i=1}^{n}$ is an open cover of $\tilde{H}$ and let $\{g_1, g_2, \dots, g_n\}$ be a partition of the unity subordinated to $\{\tilde{A}_i\}_{i=1}^n$. Further, let:

$$f(\tilde{x}) = \sum_{i=1}^{n} g_i(\tilde{x})\tilde{x}_i, \quad \tilde{x}_i = \phi(x_i) \quad (i \in \{1,2,\dots,n\})$$

for every $\tilde{x} \in \tilde{H}$.

From the Brower fixed point theorem, it follows that there exists $\tilde{u} \in \tilde{H}$ such that $\tilde{u} = f(\tilde{u})$. Let us suppose that:

$$g_i(\tilde{u}) \neq 0. \text{ for every } i \in \{i_1, i_2, \dots, i_s\}.$$

Then $\tilde{u} \in \tilde{A}_{i_k}$, for every $k \in \{1,2,\dots,s\}$ and if $u = \phi^{-1}(\tilde{u})$ we have that $\tilde{u} \in \tilde{A}_{i_k}$, for every $k \in \{1,2,\dots,s\}$.
Since $A_i \subseteq T^{-1}(Sx_i)$ for every $i \in \{1,2,\dots,n\}$ we obtain that $u \in T^{-1}Sx_{i_k}$, $k \in \{1,2,\dots,s\}$ which implies that $x_i \in S^{-1}Tu$, $i \in \{i_1, i_2, \dots, i_s\}$. From the relation $\langle S^{-1}Tu\rangle = S^{-1}Tu$, for every $u \in K$, we have that:

$$\langle x_{i_1}, x_{i_2}, \dots, x_{i_s}\rangle \subseteq S^{-1}Tu.$$

Then:

$$u = \phi^{-1}(\tilde{u}) = \phi^{-1}\Big(\sum_{k=1}^{s} g_{i_k}(\tilde{u})\tilde{x}_i\Big) \in \phi^{-1}(\langle \tilde{x}_{i_1}, \tilde{x}_{i_2}, \dots, \tilde{x}_{i_s}\rangle)$$

$$= \langle x_{i_1}, x_{i_2}, \dots, x_{i_s}\rangle \subseteq S^{-1}Tu$$

which means that $Su \cap Tu \neq \emptyset$.

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REZIME

NEKA UOPŠTENJA BRAUDEROVIH TEOREMA O NEPOKRETNOJ TAČKI U TOPOLOŠKIM PROSTORIMA

Korišćenjem metode, koju je u radu [4] dao H. Komiya, u ovom radu su uopšteni neki rezultati E. Tarafdara iz rada [6] kao i teorema iz rada [5].