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FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN NOT NECESSARILY LOCALLY CONVEX TOPOLOGICAL VECTOR SPACES

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ABSTRACT

In this paper the notions of ψ-measure of noncompactness and (ϕ,γ)-densifying mapping are introduced. Using these notions some fixed point theorems in not necessarily locally convex topological vector spaces are proved.

1. INTRODUCTION

Recently many results from the fixed point theory in not necessarily locally convex topological vector spaces have been proved ([1], [2], [3], [4], [5], [6], [7], [8], [9]).

One of these results is a generalization of the Schauder fixed point theorem for paranormed spaces proved by Zima in [8], and we shall prove some generalizations of it. It is well known that in a Banach E space we have for every bounded subset $M \subset E$ that:

(1) $\alpha(\text{co } M) = \alpha(M)$ (co M - the convex hull of M)

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(2) $\beta(\mathrm{co}\, M) \leq \beta(M)$

where α is the Kuratowski measure of noncompactness and β is the inner Hausdorff measure of noncompactness. In the case of a paranormed space we do not have, in general, (1) and (2). Hence, we shall investigate, in the case of paranormed spaces some connections between $\alpha(\mathrm{co}\, M)$ and $\alpha(M)$ and similarly between $\beta(\mathrm{co}\, M)$ and $\beta(M)$. First, we shall give some notations, definitions and results, which will be used in the following text. All topological vector spaces in this paper will be assumed to be Hausdorff.

DEFINITION 1. *Let* E *be a vector space over the real or complex number field and* $\|\ \|^*: E \to [0,\infty)$ *so that the following conditions are satisfied:*

1. $\|x\|^* = 0 \iff x = 0$.
2. $\|-x\|^* = \|x\|^*$, *for every* $x \in E$.
3. $\|x+y\|^* \leq \|x\|^* + \|y\|^*$, *for every* $x,y \in E$.
4. *If* $\|x_n - x\|^* \to 0$ *and* $\lambda_n \to \lambda$ *then* $\|\lambda_n x_n - \lambda x\|^* \to 0$, $n\to\infty$.

Then $(E, \|\ \|^*)$ *is a paranormed space and* $\|\ \|^*$ *is a paranorm.*

In a paranormed space $(E, \|\ \|^*)$ the topology is introduced by the family $V = \{V_r\}_{r>0}$ of neighbourhoods of zero in E where:

$$V_r = \{x \mid x \in E, \|x\|^* < r\} .$$

Then E is a metrizable topological vector space.

DEFINITION 2. *Let* $(E, \|\ \|^*)$ *be a paranormed space and* K *a nonempty subset of* E. *We say that* K *has a Zima's constant* $C(K) \geq 0$ *if and only if:*

$$\|tu\|^* \leq C(K)\, t\, \|u\|^* , \text{ for every } u \in K-K \text{ and every } t \in [0,1]$$

Let us give an example of a subset $K \subset E$ and $(E, \| \|^*)$, where K has a Zima's constant $C(K)$ [2].

Let $E = S(0,1)$, where $S(0,1)$ is the space of all equivalence classes of finite real measurable functions on $[0,1]$ with measure μ and for every $\hat{x} \in S(0,1)$: (μ - the Lebegue measure):

$$\| \hat{x} \|^* = \int_0^1 \frac{|x(t)|}{1+|x(t)|} \mu(dt), \quad \{x(t)\} \in \hat{x}.$$

It is known that $S(0,1)$ is an admissible topological vector space [3] and that the convergence in the paranorm is the convergence in the measure.

Let $\alpha > 0$ and $K_\alpha = \{\hat{x} | \hat{x} \in S(0,1), |x(t)| < \alpha, t \in [0,1]\}$. Then $C(K_\alpha) = 1 + 2\alpha$ [2].

DEFINITION 3. *Let* E *be a topological vector space,* $\mathcal{U}$ *be the fundamental family of neighbourhoods of zero in* E *and* $K \subset E$ *. The set* K *is said to be of Zima's type if and only if for every* $V \in \mathcal{U}$ *there exists* $U \in \mathcal{U}$ *such that* $co(U \cap (K-K)) \subset V$.

If K is a nonempty subset of E, where $(E, \| \|^*)$ is a paranormed space, and K has Zima's constant $C(K) > 0$, then for every $r > 0$.

$$co(V_{\frac{r}{C(K)}} \cap (K-K)) \subseteq V_r$$

and hence, K is of Zima's type.

By $\mathcal{R}(K)$, $K \subseteq E$ and E is a topological vector space, we shall denote the family of all nonempty, closed and convex subsets of K.

In [3] S.Hahn introduced the notion of an σ-admissible subset of a topological vector space and proved a fixed point theorem for multivalued mappings defined on σ-admissible subsets.

DEFINITION 4. *Let* E *be a topological vector space,* Z *be a nonempty closed subset of* E *and* $\sigma(Z)$ *a nonempty* *system of subsets of* Z. *The set* Z *is said to be* σ*-admissible if for each compact mapping* $F: A \to \sigma(Z)$, *where* A *is a topological space, and for each neighbourhood* V *of zero in* E, *there exists a finite dimensional vector subspace* E_V *of* E *and a compact mapping* $F_V: A \to \sigma(Z)$ *such that*

(i) $F_V(A) \subseteq E_V$

(ii) *For every* $x \in E$, $F_V(x) \subseteq F(x) + V$.

If $Z = E$ then E is called a σ-admissible topological vector space and if $\sigma = \{\{x\} \mid x \in E\}$ then E is admissible. In [2] we proved that every closed and convex subset of Zima's type is $\mathcal{R}$-admissible.

If K is a nonempty closed, convex and $\mathcal{R}$-admissible subset of a topological vector space E and $F: K \to \mathcal{R}(K)$ a compact mapping (F is upper semicontinuous and $\overline{F(K)}$ is compact) then from [3] it follows that there exists $x \in K$ such that $x \in F(x)$.

2. ψ-MEASURE OF NONCOMPACTNESS

Let E be a topological vector space, $\psi: [0,\infty) \to [0,\infty)$, $\emptyset \neq \text{co } K = K \subseteq E$, $\mathcal{M}$ a family of nonempty subsets of K such that $A \in \mathcal{M} \Rightarrow \text{co} A \in \mathcal{M}$ and $\gamma: \mathcal{M} \to [0,\infty)$.

DEFINITION 5. *The mapping* γ *is said to be a* ψ*-measure of noncompactness on* K *if and only if:*

1. $\gamma(A) = 0 \Leftrightarrow \bar{A}$ *is compact* $(A \in \mathcal{M})$.
2. $\gamma(\text{co}A) \leq \psi(\gamma(A))$ $(A \in \mathcal{M})$.

The next propositions also give examples of ψ-measures of noncompactness. If E is a Banach space from (1)and (2) it follows that α and β are $\{t\}$-measures of noncompactness, which means that $\psi(t) = t$.

PROPOSITION 1. *Let* $(E, \|\ \|^*)$ *be a complete paranormed space,* K *a nonempty bounded and convex subset of* E *which has Zima's constant* C(K), M *the family* 2^K *of all nonempty subsets of* K, *and* $\gamma = \beta$ *where* $\beta(A) = \inf\{\varepsilon \mid \varepsilon > 0$, *there exists a finite* ε*-net* $\{x_1, x_2, \ldots, x_n\} \subset A$ of A $\}$ $(A \subset K)$. *Then* β *is the* $\{C(K)t\}$ *measure of noncompactness.*

P r o o f. Let us prove that for every $A \subset K$:

$$\beta(\mathrm{co}\, A) < C(K)\beta(A).$$

Let $\varepsilon > 0$ and $\{x_1, \ldots, x_n\} \subseteq A$ be a $\beta(A) + \varepsilon$-net of A. The set $\mathrm{co}\{x_1, x_2, \ldots, x_n\}$ is precompact and so there exists $B = \{u_1, u_2, \ldots, u_r\} \subseteq \mathrm{co}\{x_1, x_2, \ldots, x_n\}$ which is an ε-net of the set $\mathrm{co}\{x_1, x_2, \ldots, x_n\}$. We shall show that for every $y \in \mathrm{co}\, A$ there exists u_k $(k \in \{1, 2, \ldots, r\})$ so that:

$$\|y - u_k\|^* < C(K)\beta(A) + \varepsilon[C(K) + 1] .$$

Let $y \in \mathrm{co}\, A$. Then there exist $y_i \in A$ and $t_i > 0$ $(i \in \{1, 2, \ldots, s\})$ so that $\sum_{i=1}^{s} t_i = 1$ and $y = \sum_{i=1}^{s} t_i y_i$. Since $y_i \in A$, there exists $x_{n(i)}$ $(n(i) \in \{1, 2, \ldots, n\})$ so that:

$$\|y_i - x_{n(i)}\|^* < \beta(A) + \varepsilon .$$

Let $x = \sum_{i=1}^{s} t_i x_{n(i)}$. Then $x \in \mathrm{co}\{x_1, x_2, \ldots, x_n\}$ and so there exists $u_k \in B$ so that $\|x - u_k\|^* < \varepsilon$. Since $\|y - x\|^* < \sum_{i=1}^{s} \|t_i(y_i - x_{n(i)})\|^* < C(K)[\beta(A) + \varepsilon]$, it follows that:

$$\|y - u_k\|^* < \|y - x\|^* + \|x - u_k\|^* < C(K)[\beta(A) + \varepsilon] + \varepsilon .$$

Since ε is an arbitrary positive number it follows that $\beta(\mathrm{co}\, A) < C(K)\beta(A)$.

PROPOSITION 2. *Let* $(E, \| \|^*)$ *be a complete paranormed space,* K *a nonempty bounded and convex subset of* E *which has Zima's constant* C(K), $M = 2^K$ *and* $\gamma = \alpha$, *where:*

$$\alpha(A) = \inf\{\varepsilon \mid \varepsilon > 0, \text{ there exists a finite cover } \{B_j\}_{j\in J}$$

of A *such that* $\operatorname{diam} B_j < \varepsilon$, *for every* $j \in J\}$.

Then α *is the* $\{[C(K)]^2 t\}$ *-measure of noncompactness.*

P r o o f. Let us prove that for every $A \subseteq K$:

$$\alpha(\operatorname{co} A) \leq [C(K)]^2 \alpha(A) . \tag{3}$$

Let $\varepsilon > 0$ and $\alpha(A) < \varepsilon$. Then there exists a cover $\{B_1, B_2, \dots, B_n\}$ of A such that $\operatorname{diam}(B_i) < \varepsilon$, $i \in \{1,2,\dots,n\}$ and let $\varepsilon' > 0$ be such that $\operatorname{diam}(B_i) < \varepsilon' < \varepsilon$, $i \in \{1,2,\dots,n\}$. We shall suppose that $B_i \subseteq K$, for every $i \in \{1,2,\dots,n\}$. It is easy to see that $\operatorname{diam}(\operatorname{co} B_i) \leq C(K) \operatorname{diam}(B_i)$ for every $i \in \{1,2,\dots,n\}$. Let:

$$S = \{(\lambda_1, \lambda_2, \dots, \lambda_n) \mid \lambda_i \geq 0,\ i \in \{1,2,\dots,n\},\ \sum_{i=1}^{n} \lambda_i = 1\}$$

and for $\lambda \in S$:

$$Y_\lambda = \{x \mid x \in E,\ x = \sum_{i=1}^{n} \lambda_i x_i,\ x_i \in \operatorname{co} B_i ,\ i \in \{1,2,\dots,n\}\}.$$

Let us prove that $\operatorname{diam}(Y_\lambda) \leq [C(K)]^2 \varepsilon$.

Let $x, y \in Y_\lambda$. Then

$$x = \sum_{i=1}^{n} \lambda_i x_i, \quad y = \sum_{i=1}^{n} \lambda_i y_i$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $(x_i, y_i) \in \operatorname{co} B_i \times \operatorname{co} B_i$ for every $i \in \{1,2,\dots,n\}$. Then:

$$\|x-y\|^* = \|\sum_{i=1}^{n} \lambda_i (x_i - y_i)\|^* \leq \sum_{i=1}^{n} \|\lambda_i (x_i - y_i)\|^*$$

$$\leq C(K) \sum_{i=1}^{n} \operatorname{diam}(\operatorname{co} B_i)\lambda_i \leq C(K) \max_{i\in\{1,2,\dots,n\}} \operatorname{diam}(\operatorname{co} B_i) \leq [C(K)]^2 \varepsilon' .$$

It is easy to see that $\operatorname{co} A \subseteq \bigcup_{\lambda\in S} Y_\lambda$. For every $X \subseteq E$ and

$\eta > 0$ let $X^{\eta} = \bigcup_{x\in X} B(x,\eta)$ $(B(x,\eta) = \{z \mid z \in E, \ \|z-x\|^* < \eta\})$.

Let $\eta = \frac{\varepsilon-\varepsilon'}{3}[C(K)]^2$ and $\delta = \frac{\eta}{C(K)T}$ where

$T = \sup\{\sum_{i=1}^{n} \|x_i\|^*, \ x_i \in \mathrm{co}B_i, \ i \in \{1,2,\dots,n\}\}$. Further

for $\lambda \in \mathbb{R}^n$: $|||\lambda|||_{\mathbb{R}^n} = \max_{i\in\{1,2,\dots,n\}} |\lambda_i|$. Let us suppose that $0 \in K$ and prove the following implication:

(4) $\qquad |||\lambda-\bar{\lambda}||| < \delta \Rightarrow Y_{\lambda} \subseteq Y^{\eta}_{\bar{\lambda}}$

We shall prove that for every $x \in Y_{\lambda}$ there exists $y \in Y_{\bar{\lambda}}$, $|||\lambda-\bar{\lambda}||| < \delta$ such that $\|x-y\|^* < \eta$. Let $x = \sum_{i=1}^{n} \lambda_i x_i \in Y_{\lambda}$, $\lambda = (\lambda_1,\lambda_2,\dots,\lambda_n)$, $x_i \in \mathrm{co}\,B_i$, $i \in \{1,2,\dots,n\}$ and $\bar{\lambda} \in S$ such that $|||\lambda-\bar{\lambda}||| < \delta$. If $y = \sum_{i=1}^{n} \bar{\lambda}_i x_i$ then:

$$\|x-y\|^* = \|\sum_{i=1}^{n} (\lambda_i-\bar{\lambda}_i)x_i\|^* <$$

$$< \sum_{i=1}^{n} |\lambda_i-\bar{\lambda}_i| C(K) \|\tilde{x}_i\|^*, \quad \tilde{x}_i = \begin{cases} x_i, & \lambda_i > \bar{\lambda}_i \\ -x_i, & \lambda_i < \bar{\lambda}_i \end{cases}$$

since $0 \in K \Rightarrow x_i \in K-K$. Since $\|\tilde{x}_i\|^* = \|x_i\|^*$ it follows that:

$$\|x-y\|^* < |||\lambda-\bar{\lambda}||| \ C(K)T < \delta\, T\, C(K) = \eta$$

and so (4) is proved. Further, the set S is compact and so there exists a finite set $\{\bar{\lambda}^1, \bar{\lambda}^2, \dots, \bar{\lambda}^r\} \subseteq S$, such that for every $\lambda \in S$ there exists $i \in \{1,2,\dots,r\}$, so that $|||\bar{\lambda}^i-\lambda||| < \delta$. This implies that

$$\bigcup_{\lambda\in S} Y_{\lambda} \subseteq \bigcup_{i=1}^{r} Y^{\eta}_{\bar{\lambda}^i} \quad \text{and so} \quad \mathrm{co}\,A \subseteq \bigcup_{i=1}^{r} Y^{\eta}_{\bar{\lambda}^i} .$$

From the inequality $\mathrm{diam}(Y^{\eta}_{\bar{\lambda}^i}) < \mathrm{diam}(Y_{\bar{\lambda}^i}) + \eta$ we obtain that:

$$\mathrm{diam}(Y^{\eta}_{\bar{\lambda}^i}) < [C(K)]^2\varepsilon' + 2\cdot\frac{\varepsilon-\varepsilon'}{3}[C(K)]^2 < [C(K)]^2\varepsilon \quad .$$

Since ε is an arbitrary number such that $\varepsilon > \alpha(A)$ it follows (3).

Suppose that $0 \notin K$ and that $x \in K$. Then $0 \in K - x = \tilde{K}$ and $C(\tilde{K}) = C(K)$. From $\tilde{A} \subseteq K$ it follows that $\tilde{A} - x \subseteq \tilde{K}$ and so for $A = \tilde{A} - x$ we have that:

$$\alpha(\mathrm{co}\,\tilde{A}) = \alpha(\mathrm{co}(A + x)) = \alpha(\mathrm{co}\,A + x) =$$
$$= \alpha(\mathrm{co}\,A) \leq [C(K)]^2 \alpha(A) = [C(K)]^2 \alpha(\tilde{A})$$

and so (3) is proved also if $0 \notin K$.

3. FIXED POINT THEOREMS

First, we shall introduce the notion of a (ϕ,γ)-densifying mapping.

DEFINITION 6. *Let γ be a ψ-measure of noncompactness on a subset K of a topological vector space E, $F: K \to 2^K$ and $\phi: [0,\infty) \to [0,\infty)$. If $A \in M$ (Def.5) implies that $F(A) \in M$ and for every $A \in M$:*

$$\gamma[F(A)] \leq \phi[\gamma(A)]$$

then the mapping F is said to be (ϕ,γ)-densifying.

THEOREM 1. *Let E be a topological vector space, G a nonempty closed and convex subset of E such that every compact convex subset of G is R-admissible, $\phi: [0,\infty) \to [0,\infty)$, $\psi: [0,\infty) \to [0,\infty)$ monotone nondecreasing, $F: G \to R(G)$ an upper semicontinuous mapping and $\gamma: 2^G \to [0,\infty)$ a ψ-measure of noncompactness on G so that for every $z \in G$ and $A \subseteq G: \gamma(A \cup \{z\}) = \gamma(A)$. If the mapping F is (ϕ,γ)-densifying and:*

$$\psi \circ \phi(t) < t, \text{ for every } t \in (0,\infty)$$

then there exists $x \in G$ such that $x \in F(x)$.

P r o o f. Let $z \in G$ and:

$$\sigma = \{Y \subseteq G:\ z \in Y,\ Y = \overline{co}\, Y\ ,\ F(Y) \subseteq Y\}\ .$$

Since $G \in \sigma$ it follows that $\sigma \neq \emptyset$. Using the Zorn lemma we conclude that there exists a minimal element Z of the family σ and $Z = \overline{co}(F(Z) \cup \{z\})$. From this we have that:

$$\gamma(Z) = \gamma(\overline{co}(F(Z) \cup \{z\})) \leq \psi(\gamma(F(Z) \cup \{z\})) =$$

$$= \psi(\gamma(F(Z))) \leq \psi \circ \phi\,[\gamma(Z)]\ .$$

Suppose that $\gamma(Z) > 0$. Then $\psi \circ \phi\,[\gamma(Z)] < \gamma(Z)$ and we obtain the contradiction: $\gamma(Z) < \gamma(Z)$. From $\gamma(Z) = 0$ we obtain that Z is compact and so Z is R-admissible. Since $F_{|Z}: Z \to Z$, using Hahn's fixed point theorem, we conclude that there exists $x \in Z$ so that $x \in Fx$.

COROLLARY 1. *Let* G *be a nonempty closed and convex subset of Zima's type of a topological vector space* E *and* ϕ, ψ, F *and* γ *as in Theorem 1. Then there exists* $x \in G$, *so that* $x \in Fx$.

P r o o f. For every $A \subseteq G$ and every $U \in \mathcal{U}$ (the fundamental system of the neighbourhoods of zero in E),we have $co(U \cap (A-A)) \subseteq (U \cap (G-G))$ and so A is of Zima's type. Since every closed and convex subset of Zima's type is R-admissible, all the conditions of Theorem 1 are satisfied.

COROLLARY 2 . *Let* E *be a complete paranormed space* , G *a nonempty closed bounded and convex subset which has a Zima's constant* , $\gamma \in \{\alpha, \beta\}$, $\phi: [0,\infty) \to [0,\infty)$, $F: G \to R(G)$ *an upper-semicontinuous mapping which is* (ϕ,γ) *-densifying and:*

$$\phi(t) < \frac{t}{C(K)}\ ,\ t > 0 \quad \textit{if}\ \ \gamma = \beta$$

or

$$\phi(t) < \frac{t}{[C(K)]^2}\ ,\ t > 0,\ \textit{if}\ \gamma = \alpha$$

Then there exists $x \in G$ *so that* $x \in F(x)$.

THEOREM 2. *Let* E *be a topological vector space,* G *be a closed and convex subset of* E *such that every convex and compact subset of* G *is* $\mathcal{R}$*-admissible,* ϕ *and* ψ *nondecreasing mappings of* $[0,\infty)$ *into* $[0,\infty)$, $\gamma: 2^G \to [0,\infty)$ *a* ψ*-measure of noncompactness on* G *and* $F: G \to \mathcal{R}(G)$ *an upper semicontinuous* (ϕ,γ)*-densifying mapping. If for every sequence* $\{X_n\}_{n\in\mathbb{N}}$ $(X_n \subseteq G,\ n\in\mathbb{N})$ *such that* $X_n = \overline{co}X_n$, $n\in\mathbb{N}$ *and* $\lim_{n\to\infty}\gamma(X_n) = 0$, *we have that* $Y = \bigcap_{n\in\mathbb{N}} X_n$ *is compact and nonempty and* $\lim_{n\to\infty}[\psi\circ\phi]^n(t) = 0$, $t>0$, *then there exists* $x\in G$ *so that* $x\in F(x)$.

P r o o f. Let $X_1 = G$ and $X_{n+1} = \overline{co}\,F(X_n)\ (n\in\mathbb{N})$. Let us prove that $\lim_{n\to\infty}\gamma(X_n) = 0$. Since γ is a ψ-measure of noncommactness on G, F is (ϕ,γ)-densifying mapping and ψ and ϕ are nondecreasing, we have that:

$$(5)\qquad \gamma(X_{n+1}) = \gamma(\overline{co}F(X_n)) < \psi(\gamma(F(X_n))) <$$
$$< \psi(\phi(\gamma(X_n))) < [\psi\circ\phi]^{n-1}(\gamma(X_1)),\quad n\in\mathbb{N}.$$

Suppose that $\gamma(X_1) = 0$. Then G is an $\mathcal{R}$-admissible compact and convex subset of E and $F: G\to\mathcal{R}(G)$ is a compact mapping, which implies by Hahn's theorem, the existence of an element $x\in G$ so that $x\in F(x)$. If $\gamma(X_1) > 0$, then (5) implies that $\lim_{n\to\infty}\gamma(X_n) = 0$ and so $Y = \bigcap_{n\in\mathbb{N}} X_n$ is a nonempty, compact, convex and so $\mathcal{R}$-admissible subset of G. Since $F_{|Y}: Y\to Y$ is a compact mapping, there exists $x\in Y$ so that $x\in F(x)$.

Using the method from [7] we shall prove the following theorem.

THEOREM 3. *Let* E *be a topological vector space,* K *a nonempty closed and convex subset of* E, $0\in K$, $U\subseteq K$ *an open neighbourhood of zero in* E, $\mathcal{N}$ *a family of nonempty compact subsets of* K, $H: [0,1]\times cl_K(U) \to \mathcal{N}$ *an upper semicontinuous* *mapping,* $\phi, \psi : [0,\infty) \to [0,\infty)$ *nondecreasing mappings,* $\gamma : 2^K \to [0,\infty)$ *a monotone* ψ*-measure of noncompactness such that for every* $z \in G$ *and* $A \subseteq G : \gamma(A \cup \{z\}) = \gamma(A)$, *and the following conditions are satisfied :*

1. *For* $t \in [0,1]$ *and* $x \in \partial_K U$: $x \notin H(t,x)$

2. *For every* $X \subseteq cl_K U$:

$$\gamma[H([0,1] \times X)] < \phi[\gamma(X)] .$$

3. *For* $\mu > 1$ *and* $x \in \partial_K U$: $\mu x \notin H(1,x)$.

4. $t \in [0,1]$, $M \in N \Rightarrow tM \in N$.

Then there exists an upper semicontinuous $(\psi\phi, \gamma)$*-densifying mapping* $g : K \to N$ *such that for all* $x \in K$:

$$x \in g(x) \iff x \in cl_K U \text{ and } x \in H(0,x) .$$

P r o o f. Let $R = \{x \mid x \in cl_K U,\ x \in H(t,x) \text{ for some } t \in [0,1]\} \cup \{x \mid x \in cl_K U,\ x \in tH(1,x) \text{ for some } t \in [0,1]\}$. Then [7] R is a nonempty, compact set such that $R \cap \partial_K U = \emptyset$. Let $\lambda : K \to [0,1]$ be such that $\lambda(R) \subseteq \{0\}$ and $\lambda(\partial_K U) \subseteq \{1\}$. Since K is a completely regular topological space such a mapping exists. As in [7], let $g : K \to N$ be defined in the following way:

$$g(x) = \begin{cases} H(2\lambda(x), x) & , \lambda(x) < \frac{1}{2} \text{ and } x \in cl_K U \\ 2(1-\lambda(x))H(1,x), & \lambda(x) > \frac{1}{2} \text{ and } x \in cl_K U \\ \{0\} & x \notin cl_K U \end{cases} .$$

The mapping g is upper semicontinuous, and let us prove that for every $X \subseteq K$:

$$\gamma(g(X)) < \psi\phi(\gamma(X)) .$$

From the definition of mapping g, it follows that $g(X) \subseteq \overline{co}(\{0\} \cup H([0,1] \times (cl_K U \cap X)))$. If $cl_K U \cap X = \emptyset$ then:

$$\gamma(g(X)) < \gamma(\{0\}) = 0 < \psi\phi(\gamma(X)) .$$

Let us suppose that $cl_K U \cap X \neq \emptyset$. Then:

$$\gamma(g(X)) < \gamma(\overline{co}(\{0\} \cup H([0,1] \times (cl_K U \cap X))) <$$
$$< \psi(\gamma(H([0,1] \times (cl_K U \cap X)))) < \psi\phi(\gamma(cl_K U \cap X)) < \psi\phi(\gamma(X)).$$

As in [7], it follows that $x \in g(x)$ if and only if $x \in cl_K U$ and $x \in H(0,x)$.

Using Theorem 1 and Theorem 3 we obtain the following result. By 2^K_{cc} we shall denote the family of all nonempty, compact and convex subsets of K.

PROPOSITION 3. *Let* E *be a topological vector space,* K *a closed and convex subset of* E, $0 \in K$, $U \subseteq K$ *be an open neighbourhood of zero in* E, $N = 2^K_{cc}$ H, ϕ, ψ *and* γ *as in Theorem 3 and* $\psi^2\phi(t) < t$, *for every* $t > 0$. *If every compact and convex subset of* K *is R-admissible, then there exists* $x \in cl_K U$ *such that* $x \in H(0,x)$.

P r o o f. Since K is convex and $0 \in K$, it follows that $M \in N$ implies $tM \in N$ for every $t \in [0,1]$. Hence, all the conditions of Theorem 3 are satisfied and there exists an upper semicontinuous $(\psi\phi,\gamma)$-densifying mapping $g: K \to N$ such that for all $x \in K$:

(6) $x \in g(x) \Longleftrightarrow x \in cl_K U$ and $x \in H(0,x)$.

Further, from Theorem 1, since $\psi^2\phi(t) < t$, $t > 0$, it follows that there exists $x_o \in K$ such that $x_o \in g(x_o)$. From (6), it follows that $x_o \in cl_K U$ and $x_o \in H(0,x_o)$.

COROLLARY 3. *Let* E *be a complete paranormed space* K *a closed and convex subset of* E *which has Zima's constant* $C(K) > 0$, $0 \in K$, U *be an open neighbourhood of zero in* E $N = 2^K_{cc}$, H *and* ϕ *as in Theorem 3 and* $\gamma = \alpha$. *If* $[C(K)]^2\phi(t) < t$, $t > 0$ *then there exists* $x_o \in cl_K U$ *such that* $x_o \in H(0,x_o)$.

P r o o f. Measure α is a monotone $\{C(K)\ ^2t\}$ - measure of noncompactness such that $\alpha(A \cup \{z\}) = \alpha(A)$, for every $A \subset K$ and every $z \in K$. Further, every subset of K is of Zima's type since K has a Zima's constant C(K), which implies that every convex and closed subset of K is R-admissible. Hence, all the conditions of Proposition 3 are satisfied, and so there exists $x_o \in cl_K U$ such that $x_o \in H(0, x_o)$.

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REZIME

TEOREME O NEPOKRETNOJ TAČKI ZA VIŠEZNAČNA PRESLIKAVANJA U NEOBAVEZNO LOKALNO KONVEKSNIM VEKTORSKO TOPOLOŠKIM PROSTORIMA

U ovom radu su uvedeni pojmovi ψ-mere nekompaktnosti i (ϕ,γ)-kondenzujućeg operatora. Korišćenjem ovih pojmova dokazane su teoreme o nepokretnoj tački za višeznačna preslikavanja u neobavezno lokalno konveksnim vektorsko topološkim prostorima.