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RIGHT π-INVERSE SEMIGROUPS

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ABSTRACT

In this paper we introduce the concept of right (left) π-inverse semigroup. The main result is Theorem 1 which is a generalization of some results of Venkatesan [10].

A semigroup S is called a *right inverse semigroup* if every principal left ideal of S has a unique idempotent generator, [10]. Throught this paper, Z^+ will denote the set of all positive integers. A semigroup S is *regular* if for every $a \in S$ there exists $m \in Z^+$ such that $a^m \in a^m S a^m$. S is a GV-*semigroup* if S is π-regular and every regular element of S is completely regular, [4]. S is a π-*inverse* semigroup if S is π-regular and every regular element of S possesses a unique inverse, [3]. A semigroup S with zero 0 is a *nil-semigroup* if for every $a \in S$, there exists $n \in Z^+$ such that $a^n = 0$. By *nil-extension* we mean an extension by a nil-semigroup. Define an equivalence Z^+ on a π-regular semigroup S by: $aZ^+b \iff Sa^n = Sb^m$, where n and m are the smallest positive integers such that a^n and b^m are regular elements [4].

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A semigroup S is a *left (right) weakly commutative* if for every $a,b \in S$ there exists $n \in Z^+$ such that $(ab)^n \in bS$ $((ab)^n \in Sa)$, [8]. A semigroup S is *weakly commutative* if for every $a,b \in S$ there exists $n \in Z^+$ such that $(ab)^n \in bSa$, [11].

LEMMA 1. *The following conditions are equivalent for an element* a *of a semigroup* S:

(i) a *is a* π*-regular;*

(ii) *There exists* $m \in Z^+$ *such that* $R(a^m)$ $(L(a^m))$ *has an idempotent generator;*

(iii) *There exists* $m \in Z^+$ *such that* $R(a^m)$ $(L(a^m))$ *has a left (right) identity.*

PROOF: (i) => (ii). This is Proposition 3.1 [2].

(i) => (iii). Let $a^m = a^m x a^m$ for some $x \in S$ and $m \in Z^+$. Then for an arbitrary $b \in R(a^m)$ there exists $y \in S$ such that $b = a^m y$. So

$$a^m x \cdot b = a^m x \cdot a^m y = a^m y = b.$$

Thus $a^m x$ is a left identity of $R(a^m)$.

(iii) => (i). Let (iii) hold and let e be a left identity of $R(a^m)$. Then $e = a^m x$ for some $x \in S^1$, so $ea^m = a^m x a^m$, i.e. $a^m = a^m x a^m$ (since e is a left identity of $R(a^m)$). Thus a is π-regular. ∎

COROLLARY 1. *A semigroup* S *is* π*-regular if and only if for every* $a \in S$ *there exists* $m \in Z^+$ *such that* $R(a^m)$ $(L(a^m))$ *has a left (right) identity.* ∎

LEMMA 2. *Let* ρ *be a congruence on a* π*-regular semigroup* S *and* $A,B \in S/\rho$ *such that* $A = ABA$, $B = BAB$. *Then there exist* $a,b \in S$ *such that* $a \in A$, $b \in B$ *and* $a = aba$, $b = bab$.

PROOF: Let $x \in A$, $y \in B$ and let $(xy)'$ be an inverse of $(xy)^{2m}$ for some $m \in Z^+$. Assume that $a = xy(xy)' \cdot (xy)^{2m-1}x$ and $b = y(xy)'(xy)^{2m-1}$. Then $a = aba$, $b = bab$.

Since $A = ABA$, $B = BAB$ we have that $x \rho xyx$ and $y \rho yxy$, so

(1) $$xy \rho (xy)^k \quad , \qquad k \in Z^+.$$

From this it follows that

$$xy(xy)' \rho (xy)^{2m}(xy)', \qquad (xy)^{2m-1} x \rho (xy)^{2m} x$$

so

$$xy(xy)'(xy)^{2m-1} x \rho (xy)^{2m}(xy)'(xy)^{2m} x = (xy)^{2m} x \ ,$$

and, since $(xy)^{2m} x \rho x$, (by (1)) we have that $a \rho x$, and similarly $b \rho y$, Thus $a \in A$, $b \in B$. ■

COROLLARY 2. *Let* ρ *be a congruence on a* π*-regular semigroup* S. *Then every* ρ*-class which is an idempotent in* S/ρ *contains an idempotent in* S.

PROOF: Let E be an idempotent in S/ρ. Since $E = EEE$ in S/ρ we have that there exist $a, a' \in E$ such that $a = aa's$, $a' = a'aa'$. Then $aa' \in EE = E$ and aa' is an idempotent.

PROPOSITION 1. *Let* ρ *be a congruence on a* π*-regular semigroup* S *and* $n \in Z^+$. *If* $A, B_1, B_2, \dots, B_n \in S/\rho$ *and* $A = AB_iA$, $B_i = B_iAB_i$, $i = 1,2,\dots,n$ *then there exist* $a, b_1, b_2, \dots, b_n \in S$ *such that* $a \in A$, $b_i \in B_i$ *and* $a = ab_ia$, $b_i = b_iab_i$, $i = 1,2,\dots,n$.

PROOF: (By induction). The statement is true for $n = 1$ by Lemma 2. Suppose that all assertion are true for some $k < n$. Then there exist $x, y_1, x_2, \dots, y_k \in S$, such that $x \in A$, $y_i \in B_i$ and $x = xy_ix$, $y_i = y_ixy_i$, $i = 1,2,\dots,k$. Assume $y_{k+1} \in B_{k+1}$. Since S is π-regular, we have that there exists $m \in Z^+$ such that $(xy_{k+1})^{2m}$ is a regular element. Let $(xy_{k+1})'$ be an inverse of $(xy_{k+1})^{2m}$.

Let us put

$$u = xy_{k+1}(xy_{k+1})'(xy_{k+1})^{2m-1} x$$

$$v_{k+1} = y_{k+1}(xy_{k+1})'(xy_{k+1})^{2m-1}$$

$$v_i = y_i xy_{k+1}(xy_{k+1})'(xy_{k+1})^{2m-1}xy_i, \quad i=1,2,\dots,k.$$

It is quite routine to show that $u \in A$, $v_i \in B_i$, and $u = uv_iu$, $v_i = v_iuv_i$ for $i = 1,2,\dots,k+1$.

REMARK. Proposition 1 is a generalization of one result of T.E. Hall, [7].

DEFINITION. *A semigroup* S *is right (left)* π*-inverse if* S *is* π*-regular and for every* $a,x,y \in S$, $a = axa = aya$ *implies* $xa = ya$ $(ax = ay)$.

THEOREM 1. *The following conditions are equivalent on a semigroup* S*:*

(i) S *is right* π*-inverse;*

(ii) S *is* π*-regular and for every* $e,f \in E(S)$ *there exists* $m \in Z^+$ *such that* $(ef)^m = (fef)^m$;

(iii) *For every* $a \in S$ *there exists* $m \in Z^+$ *such that* $L(a^m)$ *has a unique idempotent generator;*

(iv) *For every* $a \in S$ *there exists* $m \in Z^+$ *such that* $L(a^m)$ *has a unique right identity;*

(v) S *is* π*-regular and every* L^**-class contains exactly one idempotent;*

(vi) S *is* π*-regular and for every* $e,f \in E(S)$ *there exists* $m \in Z^+$ *such that* $(ef)^m \mathcal{R} (fe)^m$.

PROOF: (i) => (ii). Let $e,f \in E(S)$ and let a be an inverse of $(ef)^m$ for some $m \in Z^+$. Then from

$$(ef)^m = (ef)^m a(ef)^m = (ef)^m fa(ef)^m,$$

and by hypothesis, we have

$$a(ef)^m = fa(ef)^m, \quad \text{i.e.} \quad a(ef)^m a = fa(ef)^m a, \quad \text{so}$$

(2) $$a = fa.$$

Now

(3) $$a = a(ef)^m a = a(efe)^{m-1} fa = a(efe)^{m-1} a.$$

From (3) it follows that

$$a(efe)^{m-1} = a(efe)^{m-1} a(efe)^{m-1} a(efe)^{m-1}$$

$$= a(efe)^{m-1} efa(efe)^{m-1} a(efe)^{m-1} \quad \text{(by (2))}$$

so by hypothesis we have

$$a(efe)^{m-1} a(efe)^{m-1} = efa(efe)^{m-1} a(efe)^{m-1}$$

i.e.

$$a(efe)^{m-1} = efa(efe)^{m-1} .$$

From this and (3), it follows that $a = efa$, so by(2), we obtain

(4) $$a = ea .$$

Using (2) and (4) we have

$$(ef)^m = (ef)^m a(ef)^m = (ef)^m ea(ef)^m$$

$$= (ef)^m efa(ef)^m = (ef)(ef)^m a(ef)^m$$

$$= (ef)(ef)^m$$

$$= (ef)^{m+1} .$$

From this it follows that

$$(ef)^m = (ef)^m ef(ef)^m = (ef)^m f(ef)^m$$

so

$$ef(ef)^m = f(ef)^m .$$

Thus

$$(ef)^m = (fef)^m .$$

(ii) => (i). Let $a = axa = aya$. Then $(xa \cdot ya)^m = (ya \cdot xa \cdot ya)^m$ for some $m \in Z^+$, so $xa = ya$.

(i) => (iii). Let $a^m = a^m x a^m$ for some $x \in S$ and $m \in Z^+$. Then by Lemma 1 $L(a^m)$ has an idempotent generator e. Assume $f \in E(S)$ such that $L(a^m) = Sf$. Then $Se = Sf$, so $e = yf$, $f = xe$ for some $x,y \in S$. Now $ef = (yf)f = yf = e$, $fe = f$, so $e = efe$, i.e.

$$e = efe = e(efe)e .$$

From this and hypothesis we have $fe = efee = efe$. Thus $f = fe = efe = e$.

(iii) => (iv). Let $L(a^m)$ has the unique idempotent generator e. Then by Lemma 1, $L(a^m)$ has the unique right identity e.

(iv) => (i). Let $L(a^m)$ have the unique right identity. Then a is π-regular (Lemma 1). Assume that $a = axa = aya$, then by the uniqueness of right identity, we have that $xa = ya$.

(ii) => (v). Let $e,f \in E(S)$ and eL^*f. Then $Se = Sf$, so $e = xf$, $f = ye$. Thus $ef = e$, $fe = f$ and by the hypothesis $e = (ef)^m = (fef)^m = f$. Therefore, every L^*-class contains exactly one idempotent.

(v) => (i). Let S be π-regular. Then by Proposition [4] every L^*-class contains an idempotent (see also Proposition 3.1 [2]). Assume that $a = axa = aya$. Then $Sxa = Sa = Sya$, i.a. xaL^*ya and since every L^*-class contains exactly one idempotent, we have that $xa = ya$.

(ii) => (vi). For any $e,f \in E(S)$ there exists $m,n \in Z^+$ such that

$$(efe)^m = (fe)^m, \quad (fef)^n = (ef)^n$$

so

$$(ef)^{mn}e = (fe)^{mn} , \quad (fe)^{mn}f = (ef)^{mn} .$$

Thus

$$(ef)^k \ \mathcal{R}(fe)^k$$

where $k = mn$.

(vi) => (ii). Let $e,f \in E(S)$ and $(ef)^m \mathcal{R}(fe)^m$ for some $m \in Z^+$. Then $(ef)^m n = (fe)^m$ for some $n \in S$, so

$$e(fe)^m = e(ef)^m n = (ef)^m n = (fe)^m$$

whence $(efe)^m = (fe)^m$. ■

LEMMA 3. *A homomorphic image of a right π-inverse semigroup is also a right π-inverse semigroup.*

PROOF: Let ϕ be a homomorphism of a right π-inverse semigroup S onto T. Than for any $t \in T$, there exists $a \in S$ such that $t = \phi(a)$, $a^m = a^m x a^m$ for some $m \in Z^+$ and $x \in S$. Now, $t^m = (\phi(a))^m = \phi(a^m) = \phi(a^m x a^m) = t^m \phi(x) t^m$. Assume $e',f' \in E(t)$, then by Corollary 2 there exist $e,f \in E(S)$ such that $\phi(e) = e'$, $\phi(f) = f'$ whence by Theorem 1 we obtain

$$(e'f')^n = (\phi(e)\phi(f))^n = (\phi(ef))^n = \phi(ef)^n) = \phi((fef)^n)$$

$$= \phi(fef)^n = (\phi(f)\phi(e)\phi(f))^n = (f'e'f')$$

for some $n \in Z^+$. So T is a right ϕ-inverse semigroup (Theorem 1 (ii)). ■

By Lemma 3, we have that every condition (i) - (vi) of Theorem 1 is equivalent with the following condition:

(vii) I and S/I are right π-inverse semigroup for every ideal I of S. ■

COROLLARY 3. *The following conditions are equivalent on a semigroup* S*:*

(i) S *is a semilattice of nil-extensions of right groups;*

(ii) S *is π-regular and left weakly commutative;*

(iii) S *is a GV-semigroup and for every* $e,f \in E(S)$ *there exists* $n \in Z^+$ *such that* $(ef)^n = (fef)^n$;

(iv) S *is a GV-semigroup and every* L^**-class contains exactly one idempotent;*

(v) S *is a GV-semigroup and for every* $e,f \in E(S)$ *there exist* $n \in Z^+$ *such that* $(ef)^n \mathcal{R} (fe)^n$.

PROOF: By Theorem 2.2 [1] and Theorem 1. ■

COROLLARY 4. *The following conditions are equivalent on a semigroup* S:

(i) S *is* π*-inverse;*

(ii) S *is* π*-regular and for every* $e,f \in E(S)$ *there exists* $n \in Z^+$ *such that* $(ef)^n = (fe)^n$;

(iii) S *is* π*-regular and* $a = axa = aya$ *implies* $xax = xax$;

(iv) *For every* $a \in S$ *there exist* $n \in Z^+$ *such that* $R(a^n)$ *and* $L(a^n)$ *contain a unique idempotent generator;*

(v) S *is both right and left* π*-inverse;*

(vi) S *is* π*-regular and weakly commutative.*

PROOF: By Theorem 2.3 [1], Theorem 4.6 [3], Theorem 1 and Corollary 3 ■

By Theorem 1 and Corollaries 3 and 4 that right π-inverse semigroups are natural generalization of π-inverse semigroups and also right inverse semigroups.

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REZIME

DESNO π-INVERZNE POLUGRUPE

U ovom radu uveden je pojam desno(levo) π -inverzne polugrupe. Glavni rezultat je Teorema 1 koja je generalizacija nekih rezultata Venkatesana [10].