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CLASSIFICATION OF THE SET OF THREE-VALUED SYMMETRIC MONOTONE LOGICAL FUNCTIONS

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ABSTRACT

In this paper it is proved that 66 classes of functions of the set of three-valued monotone logical functions (among 88 possible classes) contain a symmetric function. Also, in this paper it is shown that there are exactly 36843 classes of S-bases and 54981 classes of S-pivotal incomplete sets.

1. INTRODUCTION

The set of three-valued logical functions, i.e. the union of all the functions $\{f|E_3^n \to E_3\}$ for $E_3 = \{0,1,2\}$ and $n = 0,1,2,\ldots$ is denoted by P_3.

A subset F of P_3 is said to be closed, if it does not yield a function which is not in F by means of superposition (e.g. see [1]) among the functions in F.

For closed sets F and H, such that $F \subset H$ (proper inclusion), F is H-maximal if there is no closed set G, such that $F \subset G \subset H$.

A subset of functions in H is complete in H, if every

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function in H can be represented as a superposition of the elements of the set. If the number of H-maximal sets is finite, then a subset of functions is complete in H, if it is not contained in any H-maximal set ([1]).

A finite complete set of functions in H is called a base in H, if none of its subsets is complete in H.

A set of functions $\{f_1,\dots,f_s\}$ is called a pivotal set in H, iff for every function f_i $(1 \leq i \leq s)$ there exists an H-maximal set which does not contain the function f_i, and all the other functions $f_1,\dots,f_s$ are elements of this H-maximal set. From these definitions it follows that a base is a complete pivotal set of functions.

The rank of a base (pivotal set) is the number of elements of the base (pivotal set).

Let m be the number of H-maximal sets. The function f is of the class $a_1,\dots,a_m$, $a_i \in \{0,1\}$, $1 \leq i \leq m$, where $a_i = 0$, iff the function f is an element of the i-th H-maximal set $(1 \leq i \leq m)$. Classes of functions for each base determine the class of the given base. The classes of pivotal sets are defined analogously. We shall recall some notations of functions, preserving an h-ary relation $\underline{X}$. We shall denote it by the matrix, i.e. $\underline{X}^t \subset E_3^h$. Then for n-ary vectors $a_1,\dots,a_h (a_i \in E_3^n)$,

$$\begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_h \end{pmatrix} \in \underline{X} \iff \text{for all } i, \begin{pmatrix} a_{1i} \\ \cdot \\ \cdot \\ \cdot \\ a_{hi} \end{pmatrix} \in \underline{X} .$$

Then the set of functions preserving $\underline{X}$ (denoted by Pol $\underline{X}$) is defined by

$$X = \text{Pol } \underline{X} = \left\{ f \;\middle|\; \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_h \end{pmatrix} \in \underline{X} \Rightarrow \begin{pmatrix} f(a_1) \\ \cdot \\ \cdot \\ \cdot \\ f(a_h) \end{pmatrix} \in \underline{X} \right\} .$$

THEOREM 1. ([1]) P_3 *has exactly the following 18 maximal sets:* $T_0 = \text{Pol}(0)$, $T_1 = \text{Pol}(1)$, $T_2 = \text{Pol}(2)$, $T_{01} = \text{Pol}(0\ 1)$,

$T_{12} = \mathrm{Pol}(1\ 2)$, $T_{02} = \mathrm{Pol}(0\ 2)$, $B_0 = \mathrm{Pol}\begin{pmatrix}0&1&2&1&0&2&0\\0&1&2&0&1&0&2\end{pmatrix}$

$B_1 = \mathrm{Pol}\begin{pmatrix}0&1&2&0&1&2&1\\0&1&2&1&0&1&2\end{pmatrix}$, $B_2 = \mathrm{Pol}\begin{pmatrix}0&1&2&0&2&1&2\\0&1&2&2&0&2&1\end{pmatrix}$,

$U_0 = \mathrm{Pol}\begin{pmatrix}0&1&2&1&2\\0&1&2&2&1\end{pmatrix}$, $U_1 = \mathrm{Pol}\begin{pmatrix}0&1&2&0&2\\0&1&2&2&0\end{pmatrix}$

$U_2 = \mathrm{Pol}\begin{pmatrix}0&1&2&0&1\\0&1&2&1&0\end{pmatrix}$, $M_1 = \mathrm{Pol}\begin{pmatrix}0&1&2&0&0&1\\0&1&2&1&2&2\end{pmatrix}$,

$M_0 = \mathrm{Pol}\begin{pmatrix}0&1&2&2&2&0\\0&1&2&0&1&1\end{pmatrix}$, $M_2 = \mathrm{Pol}\begin{pmatrix}0&1&2&1&1&2\\0&1&2&2&0&0\end{pmatrix}$

$$L = \mathrm{Pol}(\{\begin{pmatrix}a\\b\\c\end{pmatrix} \in E_3^3;\quad a + b = 2c(\mathrm{mod}\ 3)\}),$$

$$S = \mathrm{Pol}\begin{pmatrix}0&1&2\\1&2&0\end{pmatrix},\quad T = \mathrm{Pol}(\{\begin{pmatrix}a\\b\\c\end{pmatrix} \in E_3^3;\ |a,b,c| < 2\}).$$

The classes of functions in P_3 are determined in papers [4] and [8]. According to [4], there exist 418 classes but some of these are not different. So, the number of different classes of P is 406 ([8]). The complete list of classes is given in [4] (without superfluous classes) [5] and [7]. In [10] there is a complete list of 96 nonsimilar classes of functions of P_3.

The intersection $X_1 \cap \ldots \cap X_k$ will be denoted by $X_1 \ldots \ldots X_k$.

A function $f(x_1,\ldots,x_n)$ depends on variable x_i, if there exist $a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n,a_i',a_i''$ such that $f(a_1,\ldots,a_{i-1},a_i',a_{i-1},\ldots,a_n) \neq f(a_1,\ldots,a_{i-1},a_i'',a_{i+1},\ldots,a_n)$. If f does not depend on x_i, then x_i is called a fixed variable. Function f is a nondegenerate function if f depends on all the variables $x_1,\ldots,x_n$.

By x^m we denote $\underbrace{x,\ldots,x}_{m}$, $x \in \{0,1,2\}$.

For the set M_1 of monotone functions with respect to the order $0 < 1 < 2$, the following theorem is proved.

THEOREM 2 (Machida, [3]). M_1 *has exactly the following 13 maximal sets:*

1. $T_0' = T_0M_1$ 2. $T_2' = T_2M_1$, 3. $T_{01}' = T_{01}M_1$,

4. $T_{12}' = T_{12}M_1$, 5. $T_{02}' = T_{02}M_1$, 6. $B_0' = B_0M_1$,

7. $B_1' = B_1M_1$, 8. $B_2' = B_2M_1$, 9. $U_0' = U_0M_1$,

10. $U_2' = U_2M_1$, 11. $T' = TM_1$,

12. $M_{min} = \{f \in P_3 \mid f = \min(t_1(x_1),\ldots, t_n(x_n))\}$,

13. $M_{max} = \{f \in P_3 \mid f = \max(t_1(x_1),\ldots, t_n(x_n))\}$.

$t_1(x_1),\ldots,t_n(x_n)$ *are the unary functions. Let* t_{abc} *be such a function that* $t_{abc}(0)=a, t_{abc}(1)=b, t_{abc}(2)=c$. *Then* $t_i \in M = \{t_{000}, t_{001}, t_{002}, t_{011}, t_{012}, t_{022}, t_{111}, t_{112}, t_{122}, t_{222}\}$. *The value of the min(max) is the least (biggest) value among the arguments.*

The number of nonempty classes of functions of the set M_1 is 88 ([6]). The classes are presented in Table 1. The components in the characteristic vectors and the M_1-maximal sets in Theorem 2 are in the same order.

Interchanging 0 and 2 in the definition of all the M_1--maximal sets T_0', T_2', T_{01}', T_{12}', T_{02}', B_0', B_1', B_2' , U_0' , U_2' , T', M_{min} and M_{max}, they are mapped on the sets T_2', T_0', T_{12}', T_{01}', T_{02}', B_2', B_1', B_0', U_2', U_0', T', M_{max}, M_{min}, respectively. The set M_1 is mapped into M_1. Classes of functions c_1 and c_2 are similar if c_1 can be obtained from c_2 by the above transformation of M_1- maximal sets. The number of nonsimilar classes of M_1 is 48. For the remaining 40 classes in the last column of Table 1 the corresponding similar class is given.

By using the lexicographic algorithm described in [8], and slightly modified in [5], it is obtained that P_3 has exactly 6239721 classes of bases and 4335172 classes of pivotal incomplete sets ([8]). Also, by using the algorithm , it is shown that M_1 has exactly 118744 classes of bases and 152651 classes of pivotal incomplete sets ([6]). The maximal rank of M_1-pivotal incomplete sets is 6. The unique M_1-base of rank 7 is composed of all the functions of the following classes: 81, 82, 83, 84, 85, 86, 87 ([6]).

2. SYMMETRIC FUNCTIONS

An n-ary function $f(x_1,\dots,x_n)$ is said to be symmetric iff the following equality is valid:

$$f(x_1,\dots,x_n) = f(y_1,\dots,y_n),$$

where $(y_1,\dots,y_n)$ is an arbitrary permutation of $(x_1,\dots,x_n)$.

Symmetric functions have algebraic properties which make it desirable to treat them as a separate class .

An S-base (S-pivotal set) is a base (pivotal set) which contains only symmetric functions.

It follows from the definition that the value of n-ary symmetric functions for vectors which contain the same number of 0, the same number of 1 and the same number of 2 is equal. Hence, we define

$$f[m_0,m_1,m_2] = f(0^{m_0},1^{m_1},2^{m_2}), \qquad m_0+m_1+m_2 = n.$$

The number of n-ary symmetric functions of P_3 is $3^{\binom{n+2}{2}}$ ([9]).

In [10] it is proved that 12 classes of functions (among 406 classes) do not contain a symmetric function. P_3 has exactly 5554106 classes of S-bases, and 4099601 classes of S-pivotal incomplete sets ([10]).

In paper [9] it is proved that the number of symmetric three-valued monotone logical functions is $\binom{2n+3}{n+1}$.

In this paper it is proved that 66 classes of functions of M_1 (among the possible 88 classes) contain symmetric functions. By using the lexicographic algorithm, we show that M_1 has exactly 36843 classes of S-bases and 54981 of S-pivotal incomplete sets.

3. CLASSIFICATION OF THE SYMMETRIC FUNCTIONS OF M_1

We shall consider four cases:

a) $f \in M_{min} M_{max}$

Examples of functions of the classes 33, 37, 68, 72, 76, 80, 85, 86, 87 and 88 in paper [6] are unary functions. For nonsimilar classes in the last column of Table 1, the corresponding unary functions are given. These classes contain symmetric functions, because unary functions are symmetric functions.

b) $f \in M_{min} \bar{M}_{max}$

Let $f(x_1,\dots,x_n) = \min(t_1(x_1),\dots,t_n(x_n))$.

LEMMA 1. *The symmetric n-ary* $(n > 2)$ *nondegenerate function f of the set* M_{min} *satisfies the condition:* $t_i(a) = 0 \iff t_j(a) = 0$ *for* $1 < i,j < n$.

PROOF. Let $t_i(a) = 0$. It follows from this that

$$f(2^{i-1},a,2^{n-i}) = \min(t_1(2),\dots,t_{i-1}(2),t_i(a),t_{i+1}(2),\dots$$
$$\dots, t_n(2)) = 0$$

Since function f is symmetric, we obtain $f(2^{j-1}, a, 2^{n-j}) = f(2^{i-1}, a, 2^{n-i}) = 0$, i.e. $\min(t_1(2),\dots, t_{j-1}(2), t_j(a),t_{j+1}(2), \dots, t_n(2)) = 0$. $t_j(a) = 0$ because $t_k(2) \neq 0$ $(1 < k < n)$. The proof is finished.

LEMMA 2. 2([6]). *If* $\{t'_1,\dots,t'_{n'}\} = \{t''_1,\dots,t''_{n''}\}$ and $n' > 2$, $n'' > 2$, *the functions* $\min(t'_1(x_1),\dots,t'_{n'}(x_{n'}))$ *and* $\min(t''_1(x_1),\dots,t''_{n''}(x_{n''}))$ *are in the same class.*

TABLE 1

class.	char. vector	sym	sim	class.	char. vector	sym	sim
1	0010111111111	S+	S_{3}	45	0000110110110	S-	S_{42}
2	0010111111110	S-	S_{4}	46	0000011001111	S+	
3	0001111111111	S+		47	0000001110111	S+	S_{46}
4	0001111111101	S-		48	0000110100111	S+	
5	0010011111111	S+		49	0000110100110	S-	S_{50}
6	0001011111111	S+	S_{5}	50	0000110100101	S-	
7	0010111101111	S+		51	0010001011011	S+	S_{53}
8	0010111101110	S-	S_{12}	52	0010001011010	S-	S_{54}
9	0010111101101	S-		53	0001001011011	S+	
10	0001111110111	S+	S_{7}	54	0001001011001	S-	
11	0001111110110	S-	S_{9}	55	1000110001011	S+	
12	0001111110101	S-		56	1000110001001	S-	
13	0000111111111	S+		57	0100100110011	S+	S_{55}
14	0010011011111	S+	S_{16}	58	0100100110010	S-	S_{56}
15	0010011011110	S-	S_{17}	59	0000001011111	S+	
16	0001001111111	S+		60	0000010100111	S+	
17	0001001111101	S-		61	0000011010111	S+	S_{63}
18	0010011101111	S+		62	0000011010110	S-	S_{64}
19	0001011110111	S+	S_{18}	63	0000001101111	S+	
20	0000011111111	S+		64	0000001101101	S-	
21	0000110111111	S+		65	0001001010011	S+	
22	0000111101111	S+		66	0001001010010	S+	S_{75}
23	0000111101101	S-		67	0001001010001	S+	
24	0010011001111	S+	S_{28}	68	0001001010000	S+	t_{002}
25	0010011001110	S-	S_{29}	69	1000110000011	S+	
26	0000111110111	S+	S_{22}	70	1000110000010	S+	S_{79}
27	0000111110110	S-	S_{23}	71	1000110000001	S+	
28	0001001110111	S+		72	1000110000000	S+	S_{80}
29	0001001110101	S-		73	0010001001011	S+	S_{65}
30	1010110001011	S+		74	0010001001010	S+	S_{67}
31	1010110001010	S+	S_{36}	75	0010001001001	S+	
32	1010110001001	S+		76	0010001001000	S+	S_{68}
33	1010110001000	S+	S_{37}	77	0100100100011	S+	S_{69}
34	0101100110011	S+	S_{30}	78	0100100100010	S+	S_{71}
35	0101100110010	S+	S_{32}	79	0100100100001	S+	
36	0101100110001	S+		80	0100100100000	S+	t_{011}
37	0101100110000	S+	t_{001}	81	0000001010111	S+	
38	0000011011111	S+		82	0000010000110	S+	S_{84}
39	0000001111111	S+	S_{38}	83	0000001001111	S+	S_{81}
40	0000011101111	S+		84	0000000100101	S+	
41	0000110101111	S+		85	1010000000000	S+	S_{87}
42	0000110101101	S-		86	1100100000000	S+	t_{111}
43	0000011110111	S+	S_{40}	87	0101000000000	S+	t_{000}
44	0000110110111	S+	S_{41}	88	0000000000000	S+	t_{012}

LEMMA 3. *If* $\{t_{a1c}, t_{a22}\} \subseteq \{t_1, \ldots, t_n\}$, *then the function* $\min(t_1(x_1, \ldots, t_n(x_n))$ *is not a symmetric function.*

PROOF. Let $f(x_1, \ldots, x_n) = \min(t_{a1c}(x_1), t_{a22}(x_2), \ldots)$. Then $f(1,2^{n-1}) = 1$, $f(2,1,2^{n-2}) = 2$. Hence, f is not a symmetric function.

From the above lemmas, we obtain that classes of this case containing symmetric functions are $32(\min(t_{122}(x_1), t_{122}(x_2)))$, $36(\min(t_{001}(x_1), t_{002}(x_2)))$, $67(\min(t_{002}(x_1), t_{002}(x_2)))$, $71(\min(t_{112}(x_1), t_{112}(x_2)))$, $75(\min(t_{022}(x_1), t_{022}(x_2)))$, $79(\min(t_{011}(x_1), t_{012}(x_2)))$, $84(\min(t_{012}(x_1), t_{012}(x_2)))$.

The classes 4, 9, 12, 17, 23, 29, 42, 50, 54, 56, 64 do not contain a symmetric function.

c) $f \in \bar{M}_{min} M_{max}$

Classes of functions of this case are similar to classes of functions of case b). Hence, the classes 35, 31, 74, 78, 66, 70 and 82 contain a symmetric function and the classes 2, 11, 8, 15, 27, 25, 45, 49, 52, 58 and 62 do not contain a symmetric function.

d) $f \in \bar{M}_{min} \bar{M}_{max}$

This case contains 42 classes of functions of the set M_1.

Examples of the classes 7, 28, 30, 41, 48 and 55 given in [6] are symmetric functions. By similarity, the classes 10, 24, 34, 44 and 57 also contain symmetric functions.

The first 11 M_1-maximal sets are intersections of P_3-maximal sets and the set M_1. In relation to the considered 11 sets there are 48 different classes of functions of the set M_1. Each of these classes we can supplement in two ways: we can supplement by two positions (M_{min}, M_{max}) to the classes of the set M_1 or by 7 positions to the classes of the set P_3. 17 classes (among 48 possible classes; they correspond to the classes 5, 6, 13, 18, 18, 20, 21, 38, 39, 40, 43, 46, 47, 59, 60, 81, 83) we can supplement to the classes of the set M_1 in one way only: 11 (i.e. $\bar{M}_{min}\bar{M}_{max}$). The corresponding classes of the set M_1 contain symmetric functions ([10]). Hence, these 17 classes of the set M_1 contain a symmetric function (examples are given in [10]).

Now we consider the classes 1, 3, 14, 16, 22, 26, 51, 53, 61 and 63 of M_1 in relation to the first 11 M_1-maximal sets (hence, there are 11-ary vectors). If these classes are supplemented by 00, 01 or 10 i.e. by $M_{min}M_{max}$, $M_{min}\bar{M}_{max}$ or $\bar{M}_{min}M_{max}$, then we obtain the classes of the set M_1 containing no symmetric functions (there are the classes 2, 4, 15, 17, 23, 52, 54, 62 and 64). But, there exist corresponding classes of functions of the set P_3 containing symmetric function ([10]). Hence, these 10 classes of M_1, obtained by the addition of 11 (i.e. $\bar{M}_{min}\bar{M}_{max}$), contain symmetric functions. Examples are given in [10].

Examples of symmetric functions of the classes 65 and 69 are presented in the following tables:

65	00	01	10	11	02	20	12	21	22
0	0	0	0	0	0	0	0	0	2
1	0	0	0	0	0	0	0	0	2
2	0	0	0	0	2	2	2	2	2

69	00	01	10	11	02	20	12	21	22
0	1	1	1	1	1	1	1	1	2
1	1	1	1	1	1	1	1	1	2
2	1	1	1	1	2	2	2	2	2

By similarity, the classes 73 and 77 contain a symmetric function.

Hence, the following theorem is proved:

THEOREM 3. *From 88 possible classes of functions of the set* M_1*, 66 classes contain a symmetric function.*

The classes which contain a symmetric function are denoted by S+ in the last column of Table 1,and the remaining 22 classes by S-.

4. S-BASES OF THE SET M_1

By using the algorithm given in [8], the number of classes of S-bases and S-pivotal incomplete sets is determined. The obtained data are presented in the following table:

Rank	1	2	3	4	5	6	7	total
Number of classes of S-bases of M_1	0	0	395	13852	22076	519	1	36843
Number of classes of S-pivotal incomplete sets	65	1694	15410	32570	5187	55	0	54981

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REZIME

KLASIFIKACIJA MONOTONIH SIMETRIČNIH FUNKCIJA TROZNAČNE LOGIKE

U radu je dokazano da 66 tipova funkcija (od 88 mogućih) skupa monotonih funkcija troznačne logike sadrži neku simetričnu funkciju. Takodje, pokazano je da posmatrani skup sadrži 36843 tipova S-baza i 54981 tipova S-pivotalnih nekompletnih skupova.