ZBORNIK RADOVA
Prirodno-matematičkog fakulteta
Univerziteta u Novom Sadu
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ON A CLASS OF n-GROUPS

Zoran Stojaković and Djura Paunić
Prirodno-matematički fakultet, Institut za matematiku, 21000 Novi Sad, dr Ilije Djuričića 4, Jugoslavija

ABSTRACT

In this paper n-groups satisfying the cyclic identity (C-n-groups) are considered. It is shown that every (i,j)-associative cyclic n-quasigroup, where j-i is relatively prime to n, is an n-group and then full description of C-n-groups is given.

First we give some basic definitions and notations. Other notions from the theory of n-quasigroups can be found in [1].

The sequence $x_p, x_{p+1}, \ldots, x_q$ we shall denote by x_p^q. If $p > q$, then x_p^q will be considered empty.

An n-groupoid (Q,A) is called an n-quasigroup iff the equation $A(a_1^{i-1}, x, a_{i+1}^n) = b$ has an unique solution x for every $a_1^n, b \in Q$ and every $i \in N_n = \{1,\ldots,n\}$.

An n-quasigroup (Q,A) is called (i,j)-associative

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iff the following identity holds

$$(1) \qquad A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-1}).$$

An n-quasigroup which is (i,j)-associative for all $i,j \in N_n$ is called an n-group.

An n-quasigroup (Q,A) is called m-associative iff for every $i,j \in N_n$ and every sequence x_1^{2n-1} of elements from Q which contains at most m different elements (1) holds.

In [5] cyclic n-quasigroups which represent a generalization of semi-symetric binary quasigroups were considered. A quasigroup is called semi-symmetric iff the identity $(xy)x=y$ holds.

An n-quasigroup (Q,A) is called cyclic iff it satisfies the so-called cyclic identity

$$A(A(x_1^n),\ x_1^{n-1}) = x_n.$$

Another definition of a cyclic n-quasigroup, equivalent to the preceding one, is the following:

An n-quasigroup is cyclic iff for all $x_1^n \in Q$

$$A(x_1^n) = x_{n+1} \iff A(x_{n+1},\ x_1^{n-1}) = x_n.$$

An n-group satisfying the cyclic identity will be called a C-n-group.

THEOREM 1. *Let* (Q,A) *be an* (i,j)*-associative cyclic* n*-quasigroup. Then* A *is* (i+1,j+1)*-associative* n*-quasigroup (where* (i+1, j+1) *is reduced modulo* n*).*

PROOF. Since A is (i,j)-associative the following identity holds

$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-1}).$$

1^{0} $i \neq n,\ j \neq n.$

From the cyclicity of A it follows

$$A(x_1^{i-1},\ A(x_i^{i+n-1}),\ x_{i+n}^{2n-1}) = x_{2n} \iff$$

$$\iff A(x_{2n},\ x_1^{i-1},\ A(x_i^{i+n-1}),\ x_{i+n}^{2n-2}) = x_{2n-1}$$

and

$$A(x_1^{j-1},\ A(x_j^{j+n-1}),\ x_{j+n}^{2n-1}) = x_{2n} \iff$$

$$\iff A(x_{2n},\ x_1^{j-1},\ A(x_1^{j+n-1}),\ x_{j+n}^{2n-2}) = x_{2n-1}$$

hence

$$A(x_{2n},\ x_1^{n-1},\ A(x_i^{i+n-1}),\ x_{i+n}^{2n-2}) =$$

$$= A(x_{2n},\ x_1^{j-1},\ A(x_j^{j+n-1}),\ x_{j+n}^{2n-2}),$$

i.e. A is $(i+1,j+1)$-associative.

2^{0} $i \neq n,\ j = n.$

We have

$$A(x_1^{i-1},\ A(x_i^{i+n-1}),\ x_{i+n}^{2n-1}) = x_{2n} \iff$$

$$\iff A(x_{2n},\ x_1^{i-1},\ A(x_i^{i+n-1}),\ x_{i+n}^{2n-2}) = x_{2n-1}\ .$$

$$A(x_1^{j-1},\ A(x_j^{2n-1})) = x_{2n} \iff A(x_{2n},\ x_1^{j-1}) = A(x_j^{2n-1}) \iff$$

$$\iff A(A(x_{2n},\ x_1^{j-1}),\ x_j^{2n-2}) = x_{2n-1},$$

hence

$$A(x_{2n}, x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-2}) =$$

$$= A(A(x_{2n}, x_1^{j-1}), x_j^{2n-2}),$$

i.e. A is (i+1,1)-associative.

COROLLARY 1. *If* A *is an* (i,j)-*associative cyclic* n-*quasigroup, then for every integer* m A *is* (i+m, j+m)-*associative* n-*quasigroup (where* (i+m, j+m) *is reduced modulo* n*).*

THEOREM 2. *Let* (Q,A) *be an* (i,j)-*associative cyclic* n-*quasigroup, where* j-i *is relatively prime to* n. *Then* A *is an* n-*group*.

PROOF. In this proof all numbers are reduced modulo n. Let j-i = k. A is (i,i+k)-associative, so by the corollary of the preceding theorem A is (i+k, i+2k) associative. Repeating the process we get that A is (i+mk, i+(m+1)k)-associative for all integers m.

This means that A is (i, i+mk)-associative for all integers m. But, for every t, i+mk = t has a solution m. For, mk = t-i, and since k is relatively prime to n, k generates the group of integers modulo n.

THEOREM 3. *Let* (Q,A) *be an* n-*group, where* n = 2k, k ∈ N. *The* n-*group* (Q,A) *is a* C-n-*group iff there exists an abelian group* (Q,+) *such that* x = - x *for all* x ∈ Q *and*

$$A(x_1^n) = \sum_1^n x_i + c$$

where c *is a fixed element from* Q.

PROOF. Let (Q,A) be a C-n-group. By the Hosszú-Gluskin theorem ([2], [3]), there exist a binary group (Q,·), its automorphism θ and an element $c \in Q$, such that

$$A(x_1^n) = x_1 \theta x_2 \theta^2 x_3 \ldots \theta^{n-1} x_n c,$$

where $\theta c = c$, and for every $x \in Q$, $\theta^{n-1} x = c x c^{-1}$. Then from the cyclicity of A if follows

$$(2) \qquad x_1 \theta x_2 \theta^2 x_3 \ldots \theta^{n-1} x_n c \theta x_1 \theta^2 x_2 \ldots \theta^{n-1} x_{n-1} c = x_n.$$

From this equation putting $x_i = e$, $i = 1,\ldots, n$, where e is the unit of the group $(Q,\cdot)$, we get $c^2 = e$. If we denote $\theta x_1 \theta^2 x_2 \ldots \theta^{n-1} x_{n-1} = y$, from (2) we get

$$(3) \qquad \theta^{-1} y \, \theta^{n-1} x_n c y c = x_n.$$

Putting in (3) $x_n = c$ gives $\theta^{-1} y y c = c$ and $\theta y = y^{-1}$. From the definition of y it follows that y can take any value from Q. Since n is even, from (3) we get

$$y^{-1} x_n^{-1} c y c = x_n,$$

where from for $y = e$ we have $x_n^{-1} = x_n$ for all $x_n \in Q$. A group in which $x^2 = e$ for all x must be necessarily abelian, therefore $(Q,\cdot)$ is an abelian group.

Hence

$$A(x_1^n) = x_1 x_2^{-1} x_3 \ldots x_n^{-1} c.$$

The converse part of the theorem follows by a straightforward computation, which completes the proof of the theorem.

Since every finite group $(Q,\cdot)$ such that $x^2 = e$ for all $x \in Q$, is of order 2^t, $t \in N$, and for every $t \in N$ there exists such group, (it is $C_2 \times \ldots \times C_2$ (t - times)), we have the following corollary:

COROLLARY 2. *There exists a nontrivial* finite C-n-group* (Q,A) *of order* q, *where* n *is even, iff* $q = 2^t$, $t \in N$. *Then the binary group from Theorem 3 is isomorphic to the direct product of* t *cyclic groups of order 2.*

THEOREM 4. *Let* (Q,A) *be an* n*-group, where* $n = 2k+1$ $k \in N$. *The group* (Q,A) *is a* C-n*-group iff there exists an abelian group* (Q,+) *such that*

$$A(x_1^n) = x_1 - x_2 + x_3 - \dots + x_n + c,$$

where $c = -c$ *is an element from* Q.

PROOF. If (Q,A) is a C-n-group, then, by a similar procedure as it done in the preceding theorem, we obtain that there exist a binary group $(Q,\cdot)$ and element $c \in Q$, $c = c^{-1}$, such that

$$A(x_1^n) = x_1 x_2^{-1} x_3 \dots x_{n-1}^{-1} x_n c. \tag{4}$$

In this case the equation (3) gives

$$y^{-1} x_n c y c = x_n,$$

that is

$$y^{-1} x_n c y = x_n c,$$

hence

$$zy = yz,$$

where $x_n c = z$. So, $(Q,\cdot)$ is an abelian group.

The converse part of the theorem follows directly

* *An n-quasigroup (Q,A) is called trivial iff* $|Q| = 1$.

from the definition of A.

Since in every group there exists at least one element which is equal to its inverse, we have:

COROLLARY 3. *A nontrivial finite* C-n-*group of order* q, *where* $n = 2k+1$, $k \in N$, *exists for every* $q \in N$, *and every such* n-*group is represented by (4).*

REMARK 1. When $n = 2k$, $k \in N$, an n-group described in Theorem 3 is an n-group with unity. A unit of that n-group is the element $c \in Q$, and there are no other units.

When $n = 2k+1$, $k \in N$, then an n-group described in Theorem 4 in the case $c = 0$ is an n-group with unity and every element of that n-group is a unity, and in the case $c \neq 0$ it is an n-group without unity.

REMARK 2. In [4] it is proved that the Hosszú-Gluskin theorem is valid for m-associative n-quasigroups, where $m > n+2$, which means that every such n-quasigroup is necessarly an n-group. Hence, the theorems analogous to Theorems 3 and 4 of the present paper can be proved for cyclic m-associative n-quasigroups, $m > n+2$.

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REZIME

O JEDNOJ KLASI n-GRUPA

U ovom radu razmatrane su n-grupe koje zadovoljavaju identitet cikličnosti (C-n-grupe). Pokazano je da je svaka (i,j)-asocijativna ciklična n-kvazigrupa, gde je j-i relativno prosto sa n, n-grupa, a zatim je dat potpun opis C-n-grupa.