ZBORNIK RADOVA
Prirodno-matematičkog fakulteta
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A NOTE ON THE CONVOLUTION OF FUNCTIONS WITH COMPATIBLE CARRIERS

Stevan Pilipović

Prirodno-matematički fakultet. Institut za matematiku, 21000 Novi Sad, dr I. Djuričića 4, Jugoslavija

ABSTRACT

In this paper the relations between the notion of $A(M_p)$--compatibility and the convolution of smooth functions which satisfy suitable conditions of growth are investigated.

1.

Let f and g be locally integrable functions with A = supp f and B = supp g. If the sets A and B are compatible ([1]) i.e.

$$(1) \qquad x_n \in A,\ y_n \in B,\ |x_n| + |y_n| \to \infty \iff |x_n + y_n| \to \infty,$$

then the convolution

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt,\ x \in R,$$

exists in $L^1_{loc}(R)$ ([1], 3.2.1 Theorem).

Let A be a subset of R such that for every sequence (x_n) from A there are a sequence $(\bar{x}_n)$ from A, $\delta>0$ and a sequence of

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positive numbers (ε_n) such that

$$(*) \qquad |x_n - \bar{x}_n| < \delta,\ n \in \mathbb{N}; \quad \bigcup_{n=1}^{\infty} [\bar{x}_n - \varepsilon_n, \bar{x}_n + \varepsilon_n] \subset \overset{\circ}{A}$$

($\overset{\circ}{A}$ denotes the interior of A and $\mathbb{N}$ is the set of natural numbers).

In this case we say that A satisfies condition (*).

Let us remark that if A is the support of a continuous function not identically equal to zero, then A satisfies (*). More generally, the support of a function of a locally bounded variation on $\mathbb{R}$ which is not identical to zero almost everywhere on R, satisfies condition (*). This follows from the fact that functions of locally bounded variations have no more than countable discontinuiteties.

In this note we shall prove that if A and B are subsets of $\mathbb{R}$ which satisfy condition (*) and if for every two smooth functions $\phi \in D_A$ and $\psi \in D_B$ (see part 4), the convolution $\phi * \psi$ is a continuous function, then the sets A and B are compatible.

If locally integrable functions f and g belong to some subspace M of $L^1_{loc}(\mathbb{R})$, there is a question whether $f*g$ belongs to M. In our investigations on the convolution in the space of generalized functions of $\mathcal{K}'\{M_p\}$-type ([4], [5]), we obtain, particularly, the following result: If f and g are functions from $L^1_{loc}(\mathbb{R})$ such that for some $p \in \mathbb{N}$. f/M_p and g/M_p, are essentially bounded functions on $\mathbb{R}$ and such that the supports of f and g are $A_{max}(M_p)$-compatible, then the convolution $f*g$ exists and represents a locally integrable function such that for some p_1, $f*g/M_{p_1} \in L^\infty(\mathbb{R})$. (Definitions of the spaces $\mathcal{K}\{M_p\}$, $\mathcal{K}'\{M_p\}$ and of the notion of $A_{max}(M_p)$-compatibility are given in [5].)

If for a locally integrable function f there exists $p \in \mathbb{N}$ such that f/M_p belongs to $L^\infty(\mathbb{R})$, we shall call this function an M_p-function.

We denote by MC the set of all the continuous functions which are M_p - functions or derivatives of the first order of M_p--functions on $\mathbb{R}$.

In this note we shall prove the following assertion: Let

A and B be subsets of R which satisfy condition (*). If for any two smooth functions from K_A and K_B (see part 5), respectively, the convolution belongs to MC, then A and B are $\mathcal{A}_{max}(M_p)$-compatible.

2.

First, we shall recall from [5] the properties of a sequence (M_p) and a set $\mathcal{A}$, which we shall assume in this article.

$(M_p(x))$ is a sequence of even continuous functions on R such that:

(2) $1 \leq M_p(x)$, $x \in R$, and $M_p(x)$ increases to infinity as $x \to \infty$, $p \in N$.

(N′). For every $p \in N$, there is $p' > p$, $p' \in N$, such that $M_p/M_{p'} \in L^1$ and $M_p(x)/M_{p'}(x) \to 0$ monotonically as $x \to \infty$; (L^1 is the space of Lebesque integrable functions.)

(3) For every $p \in N$ there are $p' \in N$ and $C_{p,p'} > 0$ such that

$$M_p^2(x) \leq C_{p,p'} M_{p'}(x) \text{ for } x > C_{p,p'}.$$

We shall denote by $\mathcal{A}$ the set of non-negative functions defined on R^+, bounded on bounded domains, directed according to the ordinary relation $\leq$ (i.e. for every f and g from $\mathcal{A}$ there is an $h \in \mathcal{A}$ such that $\max\{f(x),g(x)\} \leq h(x)$, $x \in R^+$) such that:

(A_1) If a non-negative function ϕ defined on R^+ satisfies the inequality $\phi(x) \leq \psi(x)$, $x \in R^+$, for some $\psi \in \mathcal{A}$, then $\phi \in \mathcal{A}$;

(A_2) There are $\phi \in \mathcal{A}$ and $x_0 > 0$ such that $\phi(x) \geq x$ if $x \geq x_0$;

(A_3) For every $\phi \in \mathcal{A}$, $m \in N$ and $n \in N$ there is $\psi \in \mathcal{A}$ such that $m\phi(x+n) \leq \psi(x)$, $x \in R^+$. ($N_0 = N \cup \{0\}$.)

We suppose that (M_p) and $\mathcal{A}$ satisfy the following condition:

(S) For every $p \in \mathbf{N}$ and $\phi \in \mathcal{A}$ there are $p' \in \mathbf{N}$ and $x_{p,p'} > 0$ such that

$$M_p(\phi(x)) \leq M_{p'}(x) \text{ if } x > x_{p,p'}.$$

Condition (S) implies the following one:

(4) For every $p \in \mathbf{N}$, there are $p' \in \mathbf{N}$ and $x_{p,p'} > 0$ such that

$$M_p(px) \leq M_{p'}(x) \text{ if } x > x_{p,p'}.$$

If (S) holds for (M_p) and $\mathcal{A}$, we shall denote $\mathcal{A}$ by $\mathcal{A}(M_p)$. From (4) it follows:

(5) For every $p \in \mathbf{N}$, there are p' and $x_{p,p'} > 0$ such that

$$M_p(x) \leq M_{p'}(x-t)M_{p'}(t) \text{ if } x > x_{p,p'} \text{ and } t \in \mathbf{R}.$$

As in [4], we say that sets $A,B \subset \mathbf{R}$ are $\mathcal{A}$-compatible if there exists $\phi \in \mathcal{A}(M_p)$ such that

$$x \in A, y \in B \Rightarrow |x| + |y| \leq \phi(|x+y|).$$

We denote by $\mathcal{A}_{max}(M_p)$ the set defined by

$$\mathcal{A}_{max}(M_p) = \bigcup_{\mathcal{A} \in \mathcal{B}} \mathcal{A}(M_p)$$

where $\mathcal{B}$ is the set of all the sets $\mathcal{A}(M_p)$ for a given sequence (M_p).

We shall also assume that (M_p) satisfies:

(B) For every $p \in \mathbf{N}$, $r \in \mathbf{N}$ and and $\varepsilon > 0$ there is $p' \in \mathbf{N}$ and $x_{p,r,p',\varepsilon} > 0$ such that $M_p^{-1}(M_r(x) \leq \varepsilon M_p^{-1}(M_{p'}(x))$ if $x > x_{p,r,p',\varepsilon}$.

Theorem 3 from [5] characterizes $\mathcal{A}_{max}(M_p)$.

3. Let Ω be a non-negative smooth even function with the support contained in $[-1,1]$ such that $\int_R \Omega(t)dt = 1$. For a fixed $\varepsilon > 0$ and $a \in R$ we put

$$\delta_\varepsilon(t-a) = \varepsilon^{-1}\Omega(\varepsilon^{-1}(t-a)),\ t \in R.$$

Theorem 2.1.1. from [1] directly implies part (i) of the following lemma:

LEMMA 1. *(i)* $(\delta_{\varepsilon_1}(t-a_1)*\delta_{\varepsilon_2}(t-a_2))(x)$ *is a smooth non-negative function such that*

$$\operatorname{supp}(\delta_{\varepsilon_1}(t-a_1)*\delta_{\varepsilon_2}(t-a_2))(x) \subset [a_1+a_2-\varepsilon_1-\varepsilon_2, a_1+a_2+\varepsilon_1+\varepsilon_2];$$

$$\int_R (\delta_{\varepsilon_1}(t-a_1)*\delta_{\varepsilon_2}(t-a\ ))(x)dx = 1.$$

(ii) *The function* $(\delta_{\varepsilon_1}(t-a_1)*\delta_{\varepsilon_2}(t-a_2))(x)$ *has a maximum not smaller than* $(\varepsilon_1 + \varepsilon_2)^{-1}$. *(In the point* $x = a_1 + a_2$.*)*

PROOF OF (ii). Since $\delta_{\varepsilon_1}(t-a_1)$ and $\delta_{\varepsilon_2}(t-a_2)$ are symmetric according to the lines $t = a_1$ and $t = a_2$, respectively, from

$$(\delta_{\varepsilon_1}(t-a_1)*\delta_{\varepsilon_2}(t-a_2))(x) = \int_{-\varepsilon_2}^{\varepsilon_2} \delta_{\varepsilon_1}(x-u-(a_1+a_2))\delta_{\varepsilon_2}(u)du,$$

we obtain that the function $(\delta_{\varepsilon_1}(t-a_1)*\delta_{\varepsilon_2}(t-a_2))(x)$ has a proper maximum at $x = a_1 + a_2$. This maximum is not smaller than $(\varepsilon_1 + \varepsilon_2)^{-1}$ because the lentgh of $\operatorname{supp}(\delta_{\varepsilon_1}(t-a_1)*\delta_\varepsilon(t-a_2))$ is not greater than $2(\varepsilon_1 + \varepsilon_2)$.

If A is an unbounded subset of R which satisfies condition (*), we denote by D_A the set of smooth functions of the form

$$\sum_{i=1}^{\infty} \delta_{\varepsilon_i}(x - a_i),\ x \in R,$$

where (a_i) is a sequence from A such that $(|a_i|)$ strictly increases to ∞, and where (ε_i) is a bounded sequence of positive numbers such that the intervals $I_i = [a_i - \varepsilon_i, a_i + \varepsilon_i]$, $i \in N$, are disjoint and contained in $\overset{0}{A}$.

THEOREM 2. *Let* A *and* B *be unbounded subsets of* R *which satisfy condition* (*). *If for any two smooth functions* $\phi \in D_A$ *and* $\psi \in D_B$, $\phi * \psi$ *is continuous, then* A *and* B *are compatible.*

REMARK 1. *If one of the sets* A *and* B *is bounded, then they are compatible* ([1]).

PROOF. We shall use the idea of the proof of Theorem 5.1. from [3] (see [5] also).

If we suppose that A and B are not compatible, this implies that there are sequences (x_n) and (y_n) from A and B, respectively, such that $|x_n| \to \infty$ and $|y_n| \to \infty$ but $|x_n + y_n| \not\to z$. Condition(*) implies that there exist $\delta > 0$ and sequences $(\bar{x}_n)$, $(\bar{y}_n)$, (ε_n), such that

$$|\bar{x}_n - x_n| < \varepsilon_n, |\bar{y}_n - y_n| < \varepsilon_n, \varepsilon_n < \delta, \; n \in N;$$

$$\bigcup_{n=1}^{\infty} [\bar{x}_n - \varepsilon_n, x_n + \varepsilon_n] \subset \overset{0}{A}, \; \bigcup_{n=1}^{\infty} [\bar{y}_n - \varepsilon_n, \bar{y}_n + \varepsilon_n] \subset \overset{0}{B}.$$

Clearly, $|\bar{x}_n| \to \infty$, $|y_n| \to \infty$ and $(\bar{z}_n)$, where $\bar{z}_n = x_n + y_n$, is a bounded sequence. Without losing on generality, we suppose that

$$n < |\bar{x}_n| < |\bar{x}_{n+1}| - 2\delta, \quad n < |\bar{y}_n| < |\bar{y}_{n+1}| - 2\delta \text{ and } \bar{z}_n \to \bar{z}.$$

Let

$$f(t) = \sum_{i=1}^{\infty} \delta_{\varepsilon_i}(t - \bar{x}), \; t \in R,$$

and

$$g = \sum_{i=1}^{\infty} \delta_{\varepsilon_i}(t - \bar{y}_i), \ t \in R.$$

These functions are from D_A and D_B, respectively.

Clearly,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (\delta_{\varepsilon_i}(t-\bar{x}_i) * \delta_{\varepsilon_j}(t-\bar{y}_j))(x) \geq$$

$$\geq \sum_{i=1}^{\infty} (\delta_{\varepsilon_i}(t-\bar{x}_i) * \delta_{\varepsilon_i}(t-\bar{y}_i))(x), \ x \in R.$$

We shall prove that the last series diverges in the point $\bar{z}$. Namely, there are two possibilities:

(i) $\inf_{i \in N} \varepsilon_i = 0$ (ii) $\inf_{i \in N} \varepsilon_i > 0$.

In case (i) from Lemma 1 (ii) it follows that the series diverges. In case (ii) this follows trivially.

Thus we have proved that $(f*g)(\bar{z})$ does not exists because if it did, it would be equal to

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (\delta_{\varepsilon_i}(t-\bar{x}_i)\delta_{\varepsilon_j}(t-\bar{y}_j))(\bar{z}).$$

4.

Let (a_i) be a strictly increasing sequence of positive numbers such that $i < a_i$, $i \in N$, and let (ε_i) be a sequence of positive numbers such that

$$a_i + \varepsilon_i < a_{i+1} - \varepsilon_{i+1}, \quad i \in N.$$

Then the following lemma holds:

LEMMA 3. *(i) For a fixed* $p \in N$

$$t \to \alpha(t) = \sum_{i=0}^{\infty} M_p(a_i)\delta_{\varepsilon_i}(t - a_i), \; t \in R,$$

belongs to MC.

(ii) A smooth function

$$t \to \beta(t) = \sum_{i=1}^{\infty} M_i(a_i)\delta_{\varepsilon_i}(t - a_i), \; t \in R,$$

does not belong to MC.

PROOF. (i) We put

$$h(x) = \int_0^x \Big(\sum_{i=1}^{\infty} M_p(a_i)\delta_{\varepsilon_i}(t-a_i)\Big)dt, \; x \in \mathbf{R}.$$

If $x < a_1 - \varepsilon_1$, $h(x) = 0$. If $a_n < x \leq a_{n+1}$, $n \in \mathbf{N}$, we have

$$|h(x)| \leq \sum_{i=1}^{n+1} M_p(a_i) \leq nM_p(a_n) + H(x-a_{n+1} + \varepsilon_{n+1})M_p(a_{n+1})$$

$$\leq (x+1)\, M_p(x+\varepsilon_{n+1}) \,. \quad \left(H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0\,. \end{cases}\right)$$

If x is sufficiently large (this means if n is sufficiently large), from (N') and (4) we have: $x < M_p(x)$ and $M_p(x+\varepsilon_{n+1}) \leq M_{p'}(x)$ for suitable $p' \in \mathbf{N}$.

Thus, from (3) we obtain that there exists $p'' \in \mathbf{N}$ such that $h(x)/M_{p''}(x) \in L^\infty(\mathbf{R})$.

(ii) Clearly β is not an M_p-function, thus we must prove that

$$R \ni x \mapsto r(x) = \int_0^x \beta(t)dt \quad \text{is not an } M_p\text{-function.}$$

If $x \in (a_n-\varepsilon_n, a_n)$, $n \in \mathbf{N}$, we have

$$r(x) \geq \frac{1}{2} M_n(a_n) \geq \frac{1}{2} M_n(x).$$

This means that for every $p \in N$, $r(x)/M_p(x)$ is unbounded on the sequence of points $(a_n - \frac{\varepsilon_n}{2})$.

If A is an unbounded subset of R which satisfies condition (*), we denote by ${}_1K_A$ the set of smooth functions of the form

$$R \ni t \to \sum_{i=1}^{\infty} M_p(a_i)\delta_{\varepsilon_i}(t-a_i), \ p \in N,$$

where (a_i) is a sequence from A such that $(|a_i|)$ strictly increases to ∞, and where (ε_i) is a bounded sequence of positive numbers such that the intervals $I_i = [a_i-\varepsilon_i, a_i+\varepsilon_i]$, $i \in N$, are disjoint and contained in $\overset{\circ}{A}$. We shall denote by ${}_2K_A$ the set of functions of the form $\delta_\varepsilon(t-a)$ where $a \in A$ and $[a-\varepsilon, a+\varepsilon] \subset \overset{\circ}{A}$. We put

$$K_A = {}_1K_A \cup {}_2K_A .$$

THEOREM 4. *Let* A *and* B *be subsets of* R *which satisfy condition* (*) . *If for any two smooth functions* $\phi \in K_A$ *and* $\psi \in K_B$ *the convolution* $\phi * \psi$ *belongs to MC, then the sets* A *and* B *are* $A_{max}(M_p)$*-compatible.*

REMARK 2. *If the convolution* $\phi*\psi$ *belongs to MC for any* $\phi \in K_A$ *and* $\psi \in K_B$, *than* $\phi*\psi$ *is a continuous function for any* $\phi \in D_A$ *and* $\psi \in D_B$. *It means that the sets* A *and* B *(given in Theorem 4) are compatible.*

REMARK 3. *If one of the sets* A *and* B *is bounded, then these sets are* $A_{max}(M_p)$*-compatible.*

PROOF OF THEOREM 4. Let us suppose that the sets A and B are not $A_{max}(M_p)$-compatible. We shall construct a function h from K_A and a function r from K_B such that $h*r$ does not belong to MC and it will be a contradiction. Our construction is similar to the one given in [5] (Theorem 5., see also [3] Theorem 5.2.).

Since we suppose that A and B are not $A_{max}(M_p)$-compatible it follows that there exist sequences (x_n) from A and (y_n) from B such that

$$(6) \qquad |x_n| + |y_n| > 2^n(1 + M_p^{-1}(M_{p_n}(|x_n + y_n|)), \quad n \in N,$$

(M_p^{-1} is the inverse function for M_p).
where $p \in N$ is fixed and (p_n) is a sequence of natural numbers such that $M_n(x) \leq M_{p_n}(x/2)$ if $x > L_n$. The existence of sequences (p_n) and (L_n) follows from (4) and B (see Theorem 3 in [5]).

Condition (6) implies that $|x_n| + |y_n| \to \infty$ and therefore, $|z_n| = |x_n + y_n| \to \infty$ if $n \to \infty$.

There are three possibilities:

$$\text{(i)} \quad |x_n| \to \infty, \ |y_n| \to \infty; \quad \text{(ii)} \quad |x_n| \to \infty, |y_n| \not\to \infty;$$

$$\text{(iii)} \quad |x_n| \not\to \infty, \ |y_n| \to \infty.$$

We first consider case (i). Since A and B satisfy condition (*) there exist sequences, $(\bar{x}_n)$ from A, $(\bar{y}_n)$ from B, $\delta > 0$ and (ε_n) with the same properties as in the proof of Theorem 2.

From (6) we have

$$|\bar{x}_n| + |\bar{y}_n| \geq |x_n| + |y_n| - 2\delta \geq$$

$$\geq 2^n(1 + M_p^{-1}(M_{p_n}(|x_n + y_n|)) - 2\delta \geq$$

$$\geq 2^n(1 + M_p^{-1}(M_{p_n}(|\bar{x}_n + \bar{y}_n| - 2\delta))) - 2\delta.$$

Since $|\bar{x}_n| \to \infty$, $|\bar{y}_n| \to \infty$, $|\bar{x}_n + \bar{y}_n| \to \infty$, if $k_n > n$, $n \in N$, we have

$$|\bar{x}_{k_n}| + |\bar{y}_{k_n}| \geq 2^{k_n}(1 + M_p^{-1}(M_{p_{k_n}}(|\bar{x}_{k_n} + \bar{y}_{k_n}| - 2\delta))) - 2\delta >$$

$$> 2^n(1 + M_p^{-1}(M_{p_n}(|\bar{x}_{k_n} + \bar{y}_{k_n}| - 2\delta))) - 2\delta.$$

This means that without losing on generality, we can suppose that sequences $(\bar{x}_n)$ and $(\bar{y}_n)$ have the following properties:

$$M_{p_n}(|\bar{x}_n + \bar{y}_n| - 2\delta) > M_n(|\bar{x}_n + \bar{y}_n|)$$

and

$$|\bar{x}_n| + |\bar{y}_n| \geq 2^n(1 + M_p^{-1}(M_n(|\bar{x}_n + \bar{y}_n|))) - 2\delta \geq$$

$$\geq M_p^{-1}(M_n(|\bar{x}_n + \bar{y}_n|)).$$

Now we put

$$h(x) = \sum_{i=1}^{\infty} M_{p'}(|\bar{x}_i|)\, \delta_{\varepsilon_i}(t - \bar{x}_i)$$

and

$$r(x) = \sum_{i=1}^{\infty} M_{p'}(|\bar{y}_i|)\delta_{\varepsilon}(t - \bar{y}_i)$$

where we choose p' such that

$$M_{p'}(|\bar{x}_n|)M_{p'}(|\bar{y}_n|) \geq M_p(|\bar{x}_n + \bar{y}_n|) \quad \text{if} \quad n \geq i_0.$$

The existence of such a p' follows from (5) (see [5], the proof of Theorem 5). These functions are from K_A and K_B, respectively, but $h*r \notin MC$. We shall prove this. We have

$$(r*h)(x) \geq \sum_{i=1}^{\infty} M_{p'}(|\bar{x}_i|)\, M_{p'}(|\bar{y}_i|)\, \delta_{\varepsilon_i}(t-\bar{x}_i)*\delta_{\varepsilon_i}(t-\bar{y}_i) \geq$$

$$\geq \sum_{i=i_0}^{\infty} M_i(|\bar{x}_i + \bar{y}_i|)\, \delta_{\varepsilon_i}(t-\bar{x}_i)*\delta_{\varepsilon_i}(t-\bar{y}_i),\ x \in R.$$

The last series is not an element from MC as it is proved in Lemma 3 (ii). Since $h*r$ is a non-negative function, it follows that this function does not belong to MC.

Case (ii) From condition (*) it follows that there exist a sequence $(\bar{x}_n)$ from A, $\bar{y}_1 \in B$, $\delta > 0$ and (ε_n) sucu that

$$|\bar{x}_n - x_n| < \varepsilon_n, \quad |\bar{y}_1 - y_1| < \delta, \quad \varepsilon_n < \delta, \; n \in \mathbf{N},$$

$$\bigcap_{n=1}^{\infty} [x_n - \varepsilon_n, \bar{x}_n + \varepsilon_n] \subset \overset{0}{A}, \quad [\bar{y}_1 - \varepsilon_1, \bar{y}_1 + \varepsilon_1] \subset \overset{0}{B},$$

$$n < |x_n| < |x_{n+1}| - 2\delta, \qquad n < |z_n| < |\bar{z}_{n+1}| - 2\delta,$$

where

$$\bar{z}_n = \bar{x}_n + \bar{y}_1, \quad n \in \mathbf{N}.$$

From (6) we obtain

$$|\bar{x}_n| + |\bar{y}_1| > |x_n| + |y_1| - 2\delta$$

$$> 2^n(1 + M_p^{-1}(M_{p_n}(|\bar{x}_n + \bar{y}_1| - 2\delta))) - 2\delta.$$

In the same way as in case (i) of this proof, we can suppose that $(\bar{x}_n)$ satisfies the following properties :

$$M_{p_n}(|\bar{x}_n + \bar{y}_1| - 2\delta) > M_n(|\bar{x}_n + \bar{y}_1|)$$

and

$$|\bar{x}_n| + |\bar{y}_1| > M_p^{-1}(M_n(|\bar{x}_n + \bar{y}_1|)).$$

We put

$$h(x) = \sum_{i=1}^{\infty} M_{p'}(|\bar{x}_i|)\delta_{\varepsilon_i}(t - \bar{x}_i), \; x \in R.$$

and

$$r(x) = M_{p'}(\bar{y}_1)\delta_{\varepsilon_1}(x - \bar{y}_1), \; x \in R.$$

In the same way as in case (i), we prove that $r(x)*h(x) \notin MC$.

Case (iii) is symmetric to (ii); thus the proof is complete.

We say that the set $A \subset R$ satisfies condition (**) if for any sequence (x_i) from A there exist a sequence $(\bar{x}_i)$, from A and bounded sequences of positive numbers (ε_i) and $(\bar{\varepsilon}_i)$ such that

$$(**) \qquad |\bar{x}_n - x_n| < \bar{\varepsilon}_n, \; n \in N; \qquad \bigcup_{n=1}^{\infty} [\bar{x}_n - \varepsilon_n, \bar{x}_n + \varepsilon_n] \subset \mathring{A};$$

$$\inf_{n \in N} \varepsilon_n = \varepsilon > 0.$$

If this condition holds for the set A then functions in K_A are M_p-functions, because

$$\sup_{\substack{n \in N \\ t \in R}} \delta_{\varepsilon_n}(t - \bar{x}_n) < \infty.$$

Thus from Theorem 4 directly follows:

THEOREM 5. *If* A *and* B *satisfy condition* (**) *and if for any two* M_p*-functions* $\phi \in K_A$ *and* $\psi \in K_B$ *the convolution* $\phi*\psi$ *is a function from MC, then the sets* A *and* B *are* $A_{max}(M_p)$*-compatible.*

If A and B are $A_{max}(M_p)$ compatible, this holds for C and D where $C \subset A$, $D \subset B$, and this also holds for sets A_ε and B_δ $\varepsilon > 0$, $\delta > 0$ (see [4] and [5]).

($A_\varepsilon = \{x | d(x,a) < \varepsilon$ for some $a \in A\}$.)

Clearly, A satisfies condition (**). From the preceding theorem we obtain:

THEOREM 6. *If for any two functions* $\phi \in K_{A_\varepsilon}$ *and* $\psi \in K_{B_\varepsilon}$, *where* $A \subset R$, $B \subset R$, $\varepsilon > 0$, *the convolution* $\phi * \psi$ *belongs to* MC, *then* A *and* B *are* $A_{max}(M_p)$*-compatible.*

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REZIME

O KONVOLUCIJI FUNKCIJA SA KOMPAKTNIM NOSAČIMA

Dokazujemo da ako f*g postoji za svako f i g sa osobinama: f,g $\in C^\infty$ supp $f \subset A$, supp $g \subset B$, f i g su odgovarajuće brzine rasta u beskonačnosti, tada A i B zadovoljavaju odgovarajući uslov kompatibilnosti.