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# SOME ESTIMATIONS FOR POLYNOMIAL COMPLEX ZEROS

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ABSTRACT

Some inequalities for simple complex zeros of a polynomial, which are localized in disjoint discs in complex plane, are given. Some of these inequalities give the estimate for approximations of zeros, obtained by the known iterative formulas with cubic convergence. The presented results have an origin in convergence analysis for simultaneous interval methods given in [9] and [10].

0. Let $Z = \{c;r\} = \{w \mid |w - c| \leq r\}$ be the disc in the complex plane with center $c \in C$ and radius $r > 0$. For the discs $Z_j = \{c_j;r_j\}$ $(j = 1,2,\dots,n)$ the following properties are valid (see, e.g. [6]):

(1) (i) (inclusivity): $Z_1 \supseteq Z_2 \iff |c_1 - c_2| \leq r_1 - r_2$.

In particular, if $Z_2 = \{c_2;0\} = c_2$, then

(2) $$c_2 \in Z_1 \iff |c_1 - c_2| \leq r_1;$$

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(ii) (disjunctivity):

(3) $$Z_1 \cap Z_2 = 0 \iff |c_1 - c_2| > r_1 + r_2;$$

(iii) (inclusive monotonocity): If f is a rational function and

$$Z_i \subset W_i \quad (i = 1,2,\ldots,n),$$

then

$$f(Z_1,\ldots,Z_n) \subset f(W_1,\ldots,W_n).$$

Especially, if $z_i \in Z_i$ $(i = 1,2,\ldots,n)$ it follows

(4) $$f(z_1,\ldots,z_n) \in f(W_1,\ldots,W_n) .$$

LEMMA 1. *If* $r/|c| \leq a < 1$, *then*

(5) $$\frac{1}{\{c;r\}} \subset \left\{\frac{1}{c};\ \frac{(a+1)r}{|c|^2 - r^2}\right\} .$$

PROOF. Since $|c| > r$, according to (2), it follows that the disc $\{c;r\}$ does not contain the origin. For this, its inverse disc, in the complex plane, exists and is given by

(6) $$\frac{1}{\{c;r\}} = \left\{\frac{\bar{c}}{|c|^2 - r^2};\ \frac{r}{|c|^2 - r^2}\right\}.$$

Starting from the given condition $r/|c| \leq a < 1$, we obtain

$$\frac{r^2}{|c|(|c|^2 - r^2)|} \leq \frac{ar}{|c|^2 - r^2}$$

or

$$\left|\frac{1}{c} - \frac{\bar{c}}{|c|^2 - r^2}\right| \leq \frac{ar}{|c|^2 - r^2} = \frac{(a+1)r}{|c|^2 - r^2} - \frac{r}{|c|^2 - r^2}.$$

In reference to (1) and (6), inclusion (5) follows from the last inequality.

1. Consider a polynomial $P$ of degree $n \geqslant 3$

$$P(z) = \prod_{j=1}^{n} (z - \xi_j)$$

with simple real or complex zeros $\xi_1, \ldots, \xi_n$. Let $Z_j = \{z_j; r_j\} = \{z \mid |z - z_j| \leqslant r_j\}$ $(j = 1, \ldots, n)$ be disjoint discs containing one and only one zero $\xi_j$, and let

$$r = \max_{1 \leqslant j \leqslant n} r_j, \quad \rho = \min_{\substack{i,j \\ i \neq j}} \{|z_i - z_j| - r_j\},$$

$$c_j = \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{1}{z_j - z_k}, \quad \eta = \frac{(n-1)r}{\rho^2}, \quad s = \max(2, (n-1)^{1/2}).$$

Introduce the rational function $z \mapsto G(z)$ by

$$G(z) = \frac{d}{dz}(\ln P(z)) = \frac{P'(z)}{P(z)}.$$

It is easy to show that

$$G(z) = \sum_{j=1}^{n} \frac{1}{z - \xi_j}. \tag{7}$$

THEOREM 1. *Let* $\xi_j \in Z_j$ $(j = 1, 2, \ldots, n)$ *and* $\rho > rs$ *holds. Then*

$$|G(z_j) - c_j| > \frac{1}{r} - \eta. \tag{8}$$

PROOF. Since $\xi_j \in Z_j = \{z_j; r_j\}$ implies $|\xi_j - z_j| < r_j$ $(j = 1, \ldots, n)$, we have

$$|G(z_j) - c_j| = \left| \frac{1}{z_j - \xi_j} + \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{1}{z_j - \xi_k} - \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{1}{z_j - z_k} \right|$$

$$> \frac{1}{|z_j - \xi_j|} - \sum_{\substack{k=1 \\ k \neq j}}^{n} \left| \frac{1}{z_j - \xi_k} - \frac{1}{z_j - z_k} \right|$$

$$> \frac{1}{r} - \sum_{\substack{k=1 \\ k\neq j}}^{n} \frac{|\xi_k - z_k|}{|z_j - \xi_k||z_j - z_k|} > \frac{1}{r} - \frac{(n-1)r}{\rho^2} = \frac{1}{r} - \eta.$$

Obviously $\frac{1}{r} - \frac{(n-1)r}{\rho^2} > 0$ and $\rho > 2r$ if $\rho > sr$. Note that the condition $\rho > 2r$ implies

$$\min_{i,j} |c_i - c_j| > \max_{i,j}(r_i + r_j)$$

so that the discs $Z_1,\ldots,Z_n$ are disjoint (according to (3)).

THEOREM 2. *Let* $b > 1$ *a real constant and* $\beta = b^2(n-1)$. *Under condition*

(9) $$\rho > b(n-1)r$$

*the inequality*

(10) $$|\xi_j - z_j^*| < \frac{\beta^2(n-1)r^3}{(\beta-1)(\beta-2)\rho^2} \quad (j = 1,\ldots,n)$$

*holds, where*

(11) $$z_j^* = z_j - \frac{1}{G(z_j) - c_j} \quad (j = 1,\ldots,n).$$

PROOF. For $z = z_j$ we obtain, from (7),

$$\xi_j \equiv z_j - \frac{1}{G(z_j) - \sum\limits_{\substack{k=1 \\ k\neq j}}^{n} \frac{1}{z_j - \xi_k}} \quad (j = 1,\ldots,n).$$

Since $\xi_k \in Z_k$ $(k = 1,\ldots,n)$, by virtue of (4), it follows

(12) $$\xi_j \in z_j - \frac{1}{G(z_j) - \sum\limits_{\substack{k=1 \\ k\neq j}}^{n} \frac{1}{z_j - Z_k}} \quad (j = 1,\ldots,n).$$

Condition (9) implies $\rho > 2r$. Hence, in regard to the conclusion from Theorem 1, the discs $Z_1,\ldots,Z_n$ are disjoint so that $z_j \notin Z_k$ $(j \neq k)$. For this, the discs $z_j - Z_k$ do not contain the origin so that the inverse discs $(z_j - Z_k)^{-1}$ exist.

The following inclusion is proved in [9]:

$$(13) \qquad \sum_{\substack{k=1\\k\neq j}}^{n} \frac{1}{z_j - Z_k} \subset \left\{ \sum_{\substack{k=1\\k\neq j}}^{n} \frac{1}{z_j - z_k} ; \frac{(n-1)r}{\rho^2} \right\} = \{c_j;\eta\} .$$

On the basis of (12) and (13), we obtain

$$(14) \qquad \xi_j \in z_j - \frac{1}{G(z_j) - \{c_j;\eta\}} = z_j - \frac{1}{\{G(z_j) - c_j;\eta\}} \qquad (j = 1,\ldots,n).$$

Using (9), in the form $r/\rho < 1/(b(n-1))$ and inequality (8), we find

$$\frac{\eta}{|G(z_j) - c_j|} < \frac{\eta}{\frac{1}{r} - \eta} = \frac{1}{\dfrac{\rho^2}{r^2(n-1)} - 1} < \frac{1}{b^2(n-1)-1} = \frac{1}{\beta - 1}.$$

Taking $a = 1/(\beta - 1) < 1$, according to Lemma 1 we can write

$$\frac{1}{\{G(z_j) - c_j;\eta\}} \subset \left\{ \frac{1}{G(z_j) - c_j} ; \frac{\frac{\beta}{\beta - 1}\eta}{|G(z_j) - c_j|^2 - \eta^2} \right\}.$$

Now, relation (14) becomes

$$\xi_j \in \left\{ z_j - \frac{1}{G(z_j) - c_j} ; \frac{\frac{\beta}{\beta-1}\eta}{|G(z_j) - c_j|^2 - \eta^2} \right\} =$$

$$= \left\{ z_j^* ; \frac{\frac{\beta}{\beta-1}\eta}{|G(z_j) - c_j|^2 - \eta^2} \right\} \qquad (j = 1,\ldots,n).$$

Hence, on the basis of (2), we obtain the corresponding inequality

$$(15) \qquad |\xi_j - z_j^*| < \frac{\frac{\beta}{\beta-1}\eta}{|G(z_j) - c_j|^2 - \eta^2} \qquad (j = 1,\dots,n).$$

Using the estimate

$$|G(z_j) - c_j|^2 - \eta^2 > |\tfrac{1}{r} - \eta|^2 - \eta^2 = \frac{1}{r^2} - \frac{2\eta}{r}$$

$$> \frac{1}{r^2}\left[1 - \frac{2}{b^2(n-1)}\right] = \frac{1}{r^2}\frac{\beta-2}{\beta},$$

we find

$$\frac{\frac{\beta}{\beta-1}\eta}{|G(z_j) - c_j|^2 - \eta^2} < \frac{\beta^2(n-1)r^3}{(\beta-1)(\beta-2)\rho^2}.$$

According to this, inequality (10) follows from (15).

REMARK 1. Relation (11) defines the iterative formula for the simultaneous determination of polynomial complex zeros (see [1], [2], [3], [4], [5]). Since $r = O(|\xi_j - z_j|)$, inequality (10) establishes that the convergence order of iterative process (11) is three and, at the same time gives the estimate of the error for new approximations of polynomial zeros. Inequality (9) may be regarded as a condition under which the iterative method (11) converges.

2. Let $Q$ be a polynomial of degree $n$ defined by

$$Q(z) = \prod_{j=1}^{n} (z - z_j),$$

where $z_1,\dots,z_n$ are the centres of discs $Z_1,\dots,Z_n$ containing the zeros $\xi_1,\dots,\xi_n$ of the polynomial $P$.

Introduce

$$(16) \qquad h_j = \frac{P(z_j)}{Q'(z_j)} = \frac{P(z_j)}{\prod\limits_{\substack{k=1\\k\neq j}}^{n} (z_j - z_k)} \qquad (j = 1,\dots,n),$$

$$H = \max_{1 \leq j \leq n} |h_j|,$$

$$w_j = \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{h_k}{z_k - z_j} \quad (j = 1,\ldots,n),$$

$$\mu = \frac{(n-1)rH}{\rho^2}.$$

**LEMMA 2.** *Let* $b > 1$ *be a real constant. Then, under the condition*

$$\rho > b(n-1)r,$$

*the inequality*

(17) $$H < \alpha r$$

*holds, where* $\alpha = \exp(1/b)$.

**PROOF.** The sequence $\nu(k) = (1 + 1/bk)^k$ $(k \in N)$ is bounded and monotonically increasing so that

$$\nu(k) < \lim_{k \to +\infty} \nu(k) = \exp(\tfrac{1}{b})$$

for each $k \in N$. By virtue of this, we have

$$|h_j| = \frac{|P(z_j)|}{|Q'(z_j)|} = |z_j - \xi_j| \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{|z_j - \xi_k|}{|z_j - z_k|}$$

$$< r_j \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{|z_j - z_k| + r_k}{|z_j - z_k|} < r(1 + \frac{r}{\rho})^{n-1} < r\left[1 + \frac{1}{b(n-1)}\right]^{n-1}$$

$$< r\exp(\tfrac{1}{b}),$$

wherefrom

$$H < \alpha r.$$

Introduce the function $b \mapsto F(b)$ by

$$F(b) = \left(\frac{1}{\exp(1/b)} - \frac{1}{b}\right)^2 - \frac{1}{4b^4} = \left(\frac{1}{\alpha} - \frac{1}{b}\right)^2 - \frac{1}{4b^4} .$$

We can find the unique real zero $b_0$ of this function as a solution of the equation

$$f(b) \equiv 2b^2 - \exp(1/b)(2b+1) = 0,$$

which follows from $F(b) = 0$. The approximating value of this zero is $b_0 \cong 2.0357236$.

**THEOREM 3.** *Assume that*

$$\rho > b(n-1)r \quad (b > b_0). \tag{18}$$

*Then*

$$|1 - w_j| > \mu \quad (j = 1,\dots,n). \tag{19}$$

**PROOF.** The function $f(b) = 2b^2 - (2b+1)\exp(1/b)$, which vanishes for $b = b_0$, is positive for $b > b_0$ so that

$$2b^2 > (2b+1)\exp(1/b) = (2b+1)\alpha. \tag{20}$$

Using (17), (18) and (20), we find

$$1 > \frac{\alpha}{b}\left(1 + \frac{1}{2b}\right) > \frac{(n-1)\alpha r}{\rho}\left(1 + \frac{r}{\rho}\right) > \frac{(n-1)H}{\rho} + \frac{(n-1)rH}{\rho^2} ,$$

that is,

$$1 - \frac{(n-1)H}{\rho} > \frac{(n-1)rH}{\rho^2} .$$

Hence

$$|1 - w_j| = |1 - \sum_{\substack{k=1\\k\neq j}}^{n} \frac{h_k}{z_k - z_j}| > 1 - \sum_{\substack{k=1\\k\neq j}}^{n} \frac{|h_k|}{|z_k - z_j|}$$

$$> 1 - \frac{(n-1)H}{\rho} > \frac{(n-1)rH}{\rho^2} = \mu.$$

**THEOREM 4.** *Under condition (18), the inequality*

$$|\xi_j - z_j^*| < \frac{(a+1)(n-1)r^3}{F(b)\rho^2} \quad (j = 1,\dots,n) \tag{21}$$

*is valid, where* $a = \alpha/(b(n-1)(b-\alpha)) < 1$ *and*

$$z_j^* = z_j - \frac{h_j}{1 - w_j} \quad (j = 1,\dots,n). \tag{22}$$

**PROOF.** We obtain from (16)

$$\xi_j \equiv z_j - \frac{h_j}{1 - \sum\limits_{\substack{k=1\\k\neq j}}^{n} \frac{h_k}{z_k - \xi_j}} \quad (j = 1,\dots,n).$$

Since $\xi_j \in Z_j$ $(j = 1,\dots,n)$, in reference to (4) it follows

$$\xi_j \in z_j - \frac{h_j}{1 - \sum\limits_{\substack{k=1\\k\neq j}}^{n} \frac{h_k}{z_k - Z_j}} \quad (j = 1,\dots,n).$$

The following inclusion is proved in [10]:

$$\sum_{\substack{k=1\\k\neq j}}^{n} \frac{h}{z_k - Z_j} \subset \left\{ \sum_{\substack{k=1\\k\neq j}}^{n} \frac{h_k}{z_k - z_j} ; \frac{(n-1)rH}{\rho^2} \right\} = \{w_i;\mu\} .$$

From the last relation we find

$$\xi_j \in z_j - \frac{h_j}{1 - \{w_j;\mu\}} = z_j - \frac{h_j}{\{1 - w_j;\mu\}} \quad (j = 1,\dots,n). \tag{23}$$

Using the estimates from the proof of Theorem 3, we obtain

$$\frac{\mu}{|1 - w_j|} < \frac{(n-1)rH/\rho^2}{1 - \frac{(n-1)H}{\rho}} < \frac{\alpha}{b(n-1)(b-\alpha)} = a.$$

From (20) the inequality $1 > \alpha/(2b(b-\alpha))$ follows so that

$$a = \frac{\alpha}{b(n-1)(b-\alpha)} < \frac{\alpha}{2b(b-\alpha)} < 1.$$

Therefore, Lemma 1 is applicable, whence

(24) $$\frac{h_j}{\{1-w_j;\mu\}} \subset \left\{\frac{h_j}{1-w_j}; \frac{(a+1)|h_j|\mu}{|1-w_j|^2 - \mu^2}\right\}.$$

Applying (17), Theorem 3 and the estimates

$$\mu = \frac{(n-1)rH}{\rho^2} < \frac{\alpha(n-1)r^2}{\rho^2} < \frac{\alpha}{b^2(n-1)} \leq \frac{\alpha}{2b^2},$$

$$|1 - w_j|^2 - \mu^2 > \left[1 - \frac{(n-1)H}{\rho}\right]^2 - \frac{\alpha^2}{4b^4} > \left(1 - \frac{\alpha}{b}\right)^2 - \frac{\alpha^2}{4b^4},$$

we find

$$\frac{|h_j|\mu}{|1 - w_j|^2 - \mu^2} < \frac{\alpha^2(n-1)r^3}{\rho^2\left[\left(1 - \frac{\alpha}{b}\right)^2 - \frac{\alpha^2}{4b^4}\right]}$$

$$< \frac{(n-1)r^3}{\rho^2} \cdot \frac{1}{\left(\frac{1}{\alpha} - \frac{1}{b}\right)^2 - \frac{1}{4b^4}} = \frac{(n-1)r^3}{F(b)\rho^2}.$$

Now, we obtain from (24)

$$\frac{h_j}{\{1-w_j;\mu\}} \subset \left\{\frac{h_j}{1-w_j}; \frac{(a+1)(n-1)r^3}{F(b)\rho^2}\right\} \quad (j = 1,\dots,n).$$

This inclusion enables us to write (23) in the form

$$\xi_j \in \{z_j - \frac{h_j}{1-w_j}; \frac{(a+1)(n-1)r^3}{F(b)\rho^2}\} = \{z_j^*; \frac{(a+1)(n-1)r^3}{F(b)\rho^2}\} \quad (j = 1,\dots,n),$$

wherefrom, according to (2)

$$|\xi_j - z_j^*| < \frac{(a+1)(n-1)r^3}{F(b)\rho^2} \quad (j = 1,\dots,n).$$

**REMARK 2.** Relation (22) defines the iterative formula for the simultaneous determination of polynomial complex zeros (see [7], [8]). The convergence order of this method is three, which can be concluded from (21). Inequality (18) may be treated as a condition for the convergence of iterative process (22).

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REZIME

NEKE OCENE ZA KOMPLEKSNE NULE POLINOMA

Date su neke nejednakosti za proste kompleksne nule polinoma, koje su u diskovima $Z_1, Z_2, \ldots, Z_n$ ($Z_i \cap Z_j = \emptyset$, $i \neq j$). Neke od ovih nejednakosti daju ocenu za aproksimacije nula, dobijene pomoću poznatih iterativnih formula za kubnu konvergenciju. Izneti rezultati dobijeni su metodom sličnoj onoj u [9] i [10].