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FOUR COUNTERFEIT COINS

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ABSTRACT

We consider the problem of ascertaining the minimum number of weighings which suffice to determine all counterfeit (heavier) coins in a set of n coins of the same appearance, given a balancescale and the information that there are exactly four heavier coins present. A procedure which is either optimal or suboptimal is constructed for an infinite set of n's. For another infinite set of n's a procedure is constructed for which the maximum number of steps differs by just two from the information-theoretical lower bound. We also consider a slightly modified problem, i. e. the case when we are given a certain number (not greater than n) of additional coins for which we know that they are all good (not counterfeit). For that case, and arbitrary n, we determine an upper bound for the maximum number of steps of an optimal procedure which differs by just four from the information-theoretical lower bound. The proofs are given by an effective construction of a procedure.

1. INTRODUCTION

Let $X=\{c_1,c_2,\ldots c_n\}$ be a set of n coins indistinguishable except that exactly m of them are slightly heavier than the rest. We suppose that all heavier (counterfeit) coins are of equal weight, and so are all light (good) coins. If λ is the weight of a light coin, then the weight of a heavy coin is less than $\frac{m+1}{m}\lambda$, so that the larger of two numerically unequal subsets of X is always the heavier.

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Given a balance scale, we want to find an optimal weighing procedure, i. e. a procedure which minimizes the maximum number of steps (weighings) which are required to identify all heavier coins. It is clear that no information is gained by balancing two numerically unequal sets. We also suppose that the scale reveals which, if either, of two subsets of X is heavier but not by how much.

Step (A,B) will mean the balancing of A against B, where A and B are two disjoint subsets of X of the same cardinality. The possible outcomes are:

(a) $A=B$ (the sets balance),

(b) $A\neq B$ (the sets do not balance).

We use the notation $A<B$, $A>B$, where < and > between two sets mean "is lighter than" and "is heavier than" respectively.

If $A\subseteq X$, $h(A)=t$ will mean that A contains exactly t heavier coins. By $|A|$ we denote the cardinality of the set A.

By $P_n^m(\ell)$ we shall denote any procedure which enables us to identify all the heavier coins, if there are exactly m of them in the set of n coins, ℓ being the maximum number of steps to be required. $P_n^m(\leqslant \ell)$ will mean a procedure for which the maximum number of steps to be required is not greater than ℓ. A procedure $P_n^m(\ell)$ is said to be optimal if no one procedure $P_n^m(r)$ exists for some $r<\ell$. We write $\mu_m(n)=\ell$ if there is an optimal procedure $P_n^m(\ell)$. A procedure $P_n^m(\ell)$ is said to be suboptimal if $\mu_m(n)=\ell-1$. It follows by information-theoretical reasonings that

$\mu_m(n) \geqslant \lceil \log_3 \binom{n}{m} \rceil$, where $\lceil x \rceil$ denotes the least integer $\geqslant x$.

Remark that for any procedure $P_n^m(\ell)$, we have the dual procedure $P_n^{n-m}(\ell)$, because by identifying m heavier coins, we also identify n-m lighter coins.

It is well known that $\mu_1(n)=\lceil \log_3 n \rceil$. For some discussion of these matters in greater detail, see [1], [2], [3], [4] and [5]. In [6] it is proved that

$$\lceil \log_3 \binom{n}{2} \rceil \leq \mu_2(n) \leq 1 + \lceil \log_3 \binom{n}{2} \rceil$$

and a corresponding procedure is constructed such that the lower bound is reached for an infinite set of n's. In [8] it is proved that for $n=3^k+3^{k-1}$, $k>1$,

$$\mu_3(n) = \lceil \log_3 \binom{n}{3} \rceil$$

and, for $n=2\cdot 3^k$, $k\geq 2$,

$$\lceil \log_3 \binom{n}{3} \rceil \leq \mu_3(n) \leq 1 + \lceil \log_3 \binom{n}{3} \rceil .$$

In this paper we have some results for the problem of four counterfeit coins.

2. RESULTS

Theorem 1. *If* $n=3^k+3^{k-1}, k\neq 3$, then

$$\lceil \log_3 \binom{n}{4} \rceil \leq \mu_4(n) \leq 1 + \lceil \log_3 \binom{n}{4} \rceil .$$

P r o o f. It is easy to check that $3^{4k-2} < \binom{3^k+3^{k-1}}{4} < 3^{4k-1}$, i.e.

$\lceil \log_3 \binom{3^k+3^{k-1}}{4} \rceil = 4k-1$, for $k\geq 4$. Now, the statement will be proved by the inductive construction of a procedure $P^4_{3^k+3^{k-1}}(\leq 4k)$, for $k\geq 1$.

For $k=1$, we have a trivial procedure $P^4_4(0)$.

Suppose that a procedure $P^4_{3^{k-1}+3^{k-2}}(\leq 4k-4)$ is constructed. Then, a procedure $P^4_{3^k+3^{k-1}}(\leq 4k)$ can be constructed as follows.

Let $X, |X|=3^k+3^{k-1}$, be partitioned into four subset, i.e. $X= A\cup B\cup C\cup D$, where $|A|=|B|=|C|=|D|= 3^{k-1}$. It is clear that $h(A)+ h(B)+ h(C)+ h(D)=4$.

Step 1. (A,B).

Step 2. (C,D).

Step 3. (A,C).

It suffices to consider seven cases ((a)-(g) below); any other possible case is quite analogous to one of these seven.

(a) If $A=B$, $C=D$, $A=C$, then $h(A)=h(B)=h(C)=h(D)=1$. We continue by the successive application of a procedure $P^1_{3^{k-1}}(k-1)$ four times, to the sets A, B, C and D independently.

(b) If $A=B$, $C=D$, $A<C$, then $h(C)=h(D)=2$. We continue by the successive application of a procedure $P^2_{3^{k-1}}(2k-2)$ two times, to the sets C and D respectively. The construction of a procedure $P^2_{3^k}(2k)$ is given in [6].

(c) $A=B$, $C<D$, $A<C$, then $h(C)=1$ and $h(D)=3$. We continue by the successive application of two independent procedures $P^1_{3^{k-1}}(k-1)$ and $P^3_{3^{k-1}+3^{k-2}}(\leq 3k-3)$ to the sets C and D' respectively, where $D'=D\cup A'$, A' being a set of good coins from A and $|A|= 3^{k-2}$. The construction of a procedure $P^3_{3^k+3^{k-1}}(\leq 3k)$ is given in [8].

(d) If $A=B$, $C<D$, $A=C$, then $h(D)=4$. Let A' A such that $|A'|=3^{k-2}$. Then, $|D\cup A'|=3^{k-1}+3^{k-2}$ and $h(D\ A')=4$. We apply a procedure $P^4_{3^{k-1}+3^{k-2}}(\leq 4k-4)$, which can be constructed by the induction hypothesis, to the set $D\cup S'$.

(e) If $A=B$, $C<D$, $A>C$, then $h(A)=h(B)=1$ and $h(D)=2$. Now, we apply a procedure $P^1_{3^{k-1}}(k-1)$ two times, to the sets A and B independently, and a procedure $P^2_{3^{k-1}}(2k-2)$ to the set D.

(f) If $A\ B$, $C<D$, $A<C$, then $h(B)=h(C)=1$ and $h(D)=2$. We continue similarly as in case (e).

In each of these six cases ((a)-(f)), all the heavier coins will be found after at most $4k-1$ steps.

(g) If $A<B$, $C<D$, $A=C$, then $h(B\ D)=4$, $h(B)\geq 1$ and $h(D)\geq 1$.

Step 4. (B,D).

(ga) If $B<D$, then $h(B)=1$ and $h(D)=3$. We continue quite similarly as in case (c).

(gb) If $B=D$, then $h(B)=h(D)=2$. We continue quite similarly as in case (b).

(gc) If $B>D$, then $h(B)=3$ and $h(D)=1$. This case is quite analogous to case (ga).

In each of these three cases, all the heavier coins will be found after at most 4k steps.

A procedure $P^4_{3^k+3^{k-1}}(\leq 4k)$ is constructed. The theorem is proved.

REMARK 1. It is easy to see that the constructed procedure, for $k \geq 4$, is in fact either an optimal or a suboptimal procedure $P^4_{3^k+3^{k-1}}(4k)$. For k=1, we have the trivial optimal procedure $P^4_4(0)$, and for k=2, the constructed procedure is in fact either an optimal or a suboptimal procedure $P^4_{12}(7)$, since in (g), instead of $P^2_{3^{k-1}}(2k-2)$ and $P^3_{3^{k-1}+3^{k-2}}(\leq 3k-3)$ we actually use the procedures $P^2_3(1)$ and $P^3_4(2)$. It is an open question whether $P^4_{12}(7)$ is an optimal procedure.

It is also an open question whether the theorem holds for k=3, i.e. for n=36.

THEOREM 2. *If* $n=2\cdot 3^k$, k_1, *then*

$$\lceil \log_3\binom{n}{4}\rceil \leq \mu_4(n) \leq 2 + \lceil \log_3\binom{n}{4}\rceil .$$

P r o o f. It is easy to check that $3^{4k-1} < \binom{2\cdot 3^k}{4} < 3^{4k}$. $\lceil \log_3\binom{2\cdot 3^k}{4}\rceil = 4k$, for $k \geq 2$. Now, the statement will be proved by the inductive construction of a procedure $P^4_{2\cdot 3^k}(\leq 4k+2)$, for $k \geq 1$.

For k=1, it is easy to construct a procedure $P^4_6(3)$, which is in fact a procedure $P^2_6(3)$. The construction of a procedure $P^2_{2\cdot 3^k}(2k+1)$ is given in [6].

Suppose that a procedure $P^4_{2\cdot 3^{k-1}}(\leq 4k-2)$ is constructed. Then a procedure $P^4_{2\cdot 3^k}(\leq 4k+2)$ can be constructed as follows.

Let X, $X = 2\cdot 3^k$, be partitioned into six subsets, i.e. $X=A\cup B\cup C\cup D\cup E\cup F$, where $|A|=|B|=|C|=|C|=|E|=|F|=3^{k-1}$. It is clear that $h(A)+h(B)+h(C)+h(D)+h(E)+h(F)=4$.

Step 1. (A,B).

Step 2. (C,D).

Step 3. (E,F).

It suffices to consider four cases ((a)-(d) below); any other case is quite analoguous to one of these four.

(a) $A<B$, $C<D$, $E<F$. We conclude that $h(B)+h(D)+h(F)=4$, $h(B)\geq 1$, $h(D)\geq 1$, $h(F)\geq 1$.

Step 4. (B,D)

(aa) If $B<D$, then $h(B)=h(F)=1$ and $h(D)=2$. We continue by the application of a procedure $P^1_{3^{k-1}}(k-1)$ two times, to the sets B and F independently, and a procedure $P^2_{3^{k-1}}(2k-2)$ to the set D. A procedure $P^2_{3^k}(2k)$ is constructed in [6].

(ab) If $B=D$, then $h(B)=h(D)=1$ and $h(F)=2$.

(ac) If $B>D$, then $h(B)=2$, $h(D)=h(F)=1$.

Cases (ab) and (ac) are quite analoguous to case (aa). In any of these three cases, all the heavier coins will be found after at most 4k steps.

(b) $A=B$, $C<D$, $E<F$.

Step 4. (A,C).

(ba) If $A<C$, then $h(C)=h(F)=1$ and $h(D)=2$. This case is quite analoguous to case (aa).

(bb) If $A>C$, then $h(A)=h(B)=h(D)=h(F)=1$. We continue by the application of a procedure $P^1_{3^{k-1}}(k-1)$ four times, to the sets A, B, D and F independently. All the heavier coins will be found after 4k steps.

(bc) If $A=C$, then $h(D\cup E\cup F)=4$, $h(D)\geq 1$ and $h(F)\geq 1$.

Step 5. (D,F).

(bca) If $D>F$, then $h(F)=1$ and $h(D)=3$. We continue by the successive application of two procedures $P^1_{3^{k-1}}(k-1)$ and $P^3_{3^{k-1}+3^{k-2}}(\leq 3k-3)$, to the sets F and $D\cup A'$

respectively, where A' A and $|A'|=3^{k-2}$. A procedure $P^{3}_{3^k+3^{k-1}}(\leq 3k)$ is constructed in [8]. All the heavier coins will be found after at most 4k+1 steps.

(bcb) If D=F, then h(D)=h(F)=2. We continue by the application of a procedure $P^{2}_{3^{k-1}}(2k-2)$ two times, to the sets D and F independently. All the heavier coins will be found after 4k+1 steps.

(bcc) If D F, then h(D)=1, h(E F)=3 and h(F)$\geq$2.

Step 6. (D,E).

Now, there are two possibilities.

(bcca) If D=E, then h(D)=h(E)=1 and h(F)=2. We continue quite similarly as in case (aa).

(bccb) If D>E, then h(D)=1 and h(F)=3. We continue quite similarly as in case (bca).

In both cases, all the heavier coins will be found after at most 4k+2 steps.

(c) A=B, C=D, E F.

Step 4. (A,C).

(ca) If A<C, then h(C)=h(D)=1 and h(F)=2.

(cb) If A>C, then h(A)=h(B)=1 and h(F)=2.

In both cases, we continue quite similarly as in case (aa). All the heavier coins will be found after at most 4k steps.

(cc) If A=C, then h(E F)=4. We continue by the application of a procedure $P^{4}_{2\cdot 3^{k-1}}(\leq 4k-2)$ to the set $E\cup F$. This procedure can be constructed by the induction hypothesis. All the heavier coins will be found after at most 4k+2 steps.

(d) A=B, C=D, E=F. We conclude that $h(A\cup C\cup E) = h(B\cup D\cup F)=2$.

Step 4. (A,C).

Step 5. (A,E).

It suffices to consider four cases ((da)-(dd) below); any other possible case is quite analoguous to one of these four.

(da) If A=C, A>E, then h(A)=h(B)=h(C)=h(D)=1.

(db) If A<C, A<E, then h(C)=h(D)=h(E)=h(F)=1.

(dc) If A=C, A<E, then h(E)=h(F)=2.

(dd) If A>C, A>D, then h(A)=H(B)=2.

Cases (da) and (db) are quite analoguous to case (bb) and cases (dc) and (dd) are quite analoguous to case (bcb). In each case, all the heavier coins will be found after at most 4k+1 steps.

A procedure $P^4_{2\cdot 3^k}(\leq 4k+2)$ is constructed. The theorem is proved.

REMARK 2. It is easy to see that the constructed proce dure, for $k\geq 3$, is in fact a procedure $P^4_{2\cdot 3^k}(4k+2)$. For k=1, we construct an optimal procedure $P^4_6(3)$ (which is at the same time an optimal procedure $P^2_6(3)$). For k=2, our construction gives a procedure $P^4_{18}(9)$, because $P^2_{3^{k-1}}(\leq 2k-2)$ in (bcca), $P^3_{3^{k-1}+3^{k-2}}(\leq 3k-3)$ in (bccb) and $P^4_{2\cdot 3^{k-1}}(\leq 4k-2)$ in (cc) become $P^2_3(1)$, $P^3_4(2)$ and $P^4_6(3)$ respectively. Since $\lceil \log_3\binom{18}{4}\rceil = 8$, it remains an open question whether $P^4_{18}(9)$ is an optimal procedure.

3. A MODIFICATION OF THE COUNTERFEIT COINS PROBLEM

Suppose that in addition to the given set $X=\{c_1,c_2,\dots,c_n\}$ containing exactly m counterfeit coins, we have at our disposal a sufficiently large number of coins for which we know that they are all good (not counterfeit). The sets involved in balancing may contain some additional good coins.

In such a modified problem we use the notation $P'^m_n(\ell)$ and $\mu'_m(n)$ instead of $P^m_n(\ell)$ and $\mu_m(n)$ respectively. It is clear that $\mu'_m(n)\leq\mu_m(n)$.

THEOREM 3. *Let* $n \geq 4$. *Then*

$$\lceil \log_3 \binom{n}{4} \rceil \leq \mu_4'(n) \leq 4 + \lceil \log_3 \binom{n}{4} \rceil .$$

P r o o f. If $n=3^k+3^{k-1}$ or $n=2\cdot 3^k$, $k \geq 1$, then the statement follows from Theorems 1 and 2.

If $2\cdot 3^{k-1}<n<3^k+3^{k-1}$, we add $3^k+3^{k-1}-n$ good coins to the set X, and obtain a set X' of 3^k+3^{k-1} coins. Now, for $k \geq 2$, we construct a procedure $P'^{4}_{3^k+3^{k-1}}(\leq 4k)$ as in Theorem 1. The statement follows since $\lceil \log_3 \binom{n}{4} \rceil \geq 4k-4$.

If 3^k+3^{k-1} n $2\cdot 3^k$, we add $2\cdot 3^k-n$ good coins to the set X, and obtain a set X' of $2\cdot 3^k$ coins. Now, for $k \geq 1$, we construct a procedure $P'^{4}_{2\cdot 3^k}(\leq 4k+2)$ as in Theorem 2. The statement follows since $\lceil \log_3 \binom{n}{4} \rceil \geq 4k-1$. (Moreover, 4 may be replaced by 3 as an upper bound).

REMARK 3. It is easy to see that we need at most n-2 additional good coins if $2\cdot 3^{k-q}<n<3^k+3^{k-1}$, and at most $\frac{n-3}{2}$ additional good coins if $3^k+3^{k-1}<n<2\cdot 3^k$, for any k.

REMARK 4. The construction of a procedure $P^3_{3^k}(3k)$ (given in [7]) seems to be more complicated than the construction of a procedure $P^3_{3^k+3^{k-1}}(\leq 3k)$. That is why in the proofs of Theorems 1 and 2, we use some coins already identified as good coins in order to enlarge the set containing three counterfeit coins. Instead of a set containing 3^{k-1} coins, we use an enlarged set containing $3^{k-1}+3^{k-2}$ coins.

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REZIME

ČETIRI NEISPRAVNA NOVČIĆA

Posmatra se problem odredjivanja minimalnog broja merenja dovoljnih za odredjivanje svih neispravnih novčića u skupu od n novčića, uz predpostavku da su u tom skupu tačno četiri neispravna. Konstruisan je jedan algoritam koji je optimalan ili skoro optimalan za jedan beskonačan skup vrednosti parametara n. Za jedan drugi beskonačan skup vrednosti od n konstruisan je algoritam za koji je maksimalan broj koraka samo za 2 veći od informaciono-teorijske donje granice. Izučavan je, takodje, nešto modifikovan problem, tj. slučaj kada se raspolaže sa izvesnim brojem novčića za koje sigurno znamo da su ispravni. Dokazi su dati efektivnom konstrukcijom odgovarajućih algoritama.