Zbornik radova Prirodno-matematičkog fakulteta-Univerzitet u Novom Sadu
knjiga 14,1 (1984)
Review of Research Faculty of Science-University of Novi Sad, Volume 14,1(1984)

A NEW KIND OF OTSUKI SPACE WITH A REGULAR METRIC CONNECTION

Nevena Pušić
Prirodno-matematički fakultet. Institut za matematiku
21000 Novi Sad, ul.dr Ilije Djuričića br.4, Jugoslavija

ABSTRACT

In the present paper, a generalized case of some properties of conformal curvature-like tensors in Weyl-Otsuki spaces of the second kind is given; especially in regard to the conformal properties of the adjoint Riemannian space, when the charactheristic tensor P is recurrent.

INTRODUCTION

In my previous paper [6], I investigated some features of so-called Weyl-Otsuki spaces of the second kind, regarding their tensor P.

I have to recall some of the properties of these spaces. As is well known, T. Otsuki developed the concept of a general connection. Γ as a cross-section over the fibre bundle $T(M) \otimes \mathcal{D}^2(M)$, over a differentiable manifold M. So, a general connection can be expressed in the form

$$(0.1) \qquad \Gamma = \partial u_j \otimes (P_i^j d^2 u^i + \Gamma_{ik}^j du^i \otimes du^k)$$

where (P_i^j) are tensor components of the type $(1,1)$ and (Γ_{ik}^j) is a generalization of an object of connection. It can

---

AMS Mathematics subject classification (1980): 53B15
Key words and phrases: Regular general connection, curvatore tensor, adjoint Riemannian space, gradient vector field, covariant metric connection, conformal curvatore tensor.

be simply an object of connection if and only if $P_i^j = \delta_i^j$ for all components of the tensor P; then such a general connection is a classical affine connection.

If the tensor (P_i^j) is regular (that means $|P_i^j| \neq 0$), the basic covariant differentiation is split up in to two classical affine connections; one of them operates on contravariant indices and the other one operates on contravariant indices of the differentiated tensor. These two connections are mutually dependent by the fact that the inverse of tensor P (it exists because of the regularity of tensor P) is covariantly constant. Otsuki expressed the coefficients of these connections in the form

$$(0.2) \qquad \frac{\partial P_j^i}{\partial u^k} + {}''\Gamma_{sk}^i P_j^s - {}'\Gamma_{jk}^s P_s^i = 0$$

The covariant part of a regular general connection is denoted by $''\Gamma$ and the contravariant one by $'\Gamma$.

A. Moor provided the Otsuki space (actually, the space with a regular general connection) with a recurrent metrics ($g_{ij,k} = \gamma_k g_{ij}$); if the pure covariant tensor P (after lowering the upper index) was symmetric as well as $''\Gamma$, these spaces were named Weyl-Otsuki spaces. They are very well investigated from all their aspect in papers [1], [2], [3].

M. Prvanović defined a Weyl-Otsuki space of the second kind as a metric Otsuki space, with a metric tensor recurrent to a symmetric covariant tensor (i.e. not obligately to itself), with a symmetric covariant tensor P and with a symmetric contravariant part of the connection ([4], [5]).

Following the same ideas in a similar way, I investigated the features of a whole class of Weyl-Otsuki spaces of the second kind, even if I named them differently. In spite of some mistakes in the most general case, I got a sequence of results about such spaces with $P = S(x)\delta$, where S was at least a C^3 real function (this case was the nearest one to Riemannian spaces), especially for their curvature and curvature-like tensors.

Now, I want to expand some of these results to a more general case.

1. I am considering the case with a characterization $\overset{o}{\nabla}_k P_{ij} = \pi_k g_{ij}$ (or $\overset{o}{\nabla}_k P^i_j = \pi_k \delta^i_j$); the previous case is actually a special case of this one, with π_k as a gradient vector field. Now, π_k is an arbitrary vector field.

In this case, according to the results of M. Prvanović in [4], we have such a form of expression for coefficients of both splits of the regular metric general connection that

$$(1.1)\qquad \begin{aligned} {}''\overset{m}{\Gamma}{}^i_{jk} &= \left\{ {i \atop jk} \right\} + \tilde{\pi}_j \delta^i_k - \tilde{\pi}^i g_{jk} \\ {}'\overset{m}{\Gamma}{}^i_{jk} &= \left\{ {i \atop jk} \right\} + \pi_j \delta^i_k + \pi_k \delta^i_j - \tilde{\tilde{\pi}}^i P^e_k P_{je} \end{aligned}$$

$''\overset{m}{\Gamma}$ denotes a classical affine connection with the vector of reccurency (γ) equal to zero. $'\overset{m}{\Gamma}$ is its complement to a regular general connection; it is not metric itself. The components of vector (π) get overbars if they are obtained from the components of the original vector (π) by applying the regular operator Q (the inverse of tensor P).

Now, we can consider some propositions of a pure covariant curvature tensor of the covariant metric connection.

PROPOSITION 1. *The curvature tensor of the covariant metric connection is skew-symmetric in the first two indices if and only if the adjoint Riemannian space is flat.*

PROPOSITION 2. *The curvature tensor of the covariant metric connection is invariant on the change of places of the first and the second pairs of indices if and only if the vector (π) is a gradient vector.*

These two propositions can be easily proved by calculating the components of the curvature tensor in terms of expression (1.1). We can easily get

$$(1.2)\qquad \overset{m}{{}''R}_{irkj} = K_{irkj} + g_{ij}\theta_{kr} - g_{ik}\theta_{jr} + g_{rk}\theta_{ji} - g_{rj}\theta_{ki}$$

where θ_{kr} denotes $\overset{o}{\nabla}_k\tilde{\pi}_r - \tilde{\pi}_r\tilde{\pi}_k + \frac{1}{2}\tilde{\pi}_s\tilde{\pi}^s g_{kr}$. It can be seen immediately that θ_{kr} can be symmetric only if $\overset{o}{\nabla}_k\tilde{\pi}_r$ is symmetric i.e. only if $\tilde{\pi}_r$ is a gradient vector of a function of at least C^3; in the more common case, the curvature tensor of this connection has not the same features as the Riemann--Christoffel tensor of a Riemannian space. But we can obtain some other features of this tensor by using formula (1.2). Denoting

$$(1.3)\qquad \overset{m}{{}''R}_{rk} = \overset{m}{{}''R}_{irkj}g^{ij}; \qquad \overset{m}{{}''R} = \overset{m}{{}''R}_{rk}g^{rk}$$

and transvecting (1.2) by the contravariant components of the metric tensor, we get, as in [4] and [6]:

$$\theta_{rk} = \frac{\overset{m}{{}''R}_{rk} - K_{rk}}{n-2} - \frac{\overset{m}{{}''R} - K}{2(n-1)(n-2)}\, g_{rk}$$

In this way, we get from (1.2)

$$(1.4)\qquad \begin{aligned} &\overset{m}{{}''R}_{irkj} + \frac{1}{2-n}(\overset{m}{{}''R}_{rk}g_{ij} + \overset{m}{{}''R}_{ij}g_{kr} - \overset{m}{{}''R}_{rj}g_{ik} - \overset{m}{{}''R}_{ik}g_{rj}) + \\ &\quad + \frac{\overset{m}{{}''R}}{(n-1)(n-2)}(g_{ij}g_{rk} - g_{ik}g_{rj}) = \\ &= K_{irkj} + \frac{1}{2-n}(K_{rk}g_{ij} + K_{ji}g_{kr} - K_{rj}g_{ik} - K_{ik}g_{rj}) + \\ &\quad + \frac{K}{(n-1)(n-2)}(g_{ij}g_{rk} - g_{ik}g_{rj}) \end{aligned}$$

So, we get an important feature:

THEOREM 1.1. *If the tensor* P *of a Weyl-Otsuki space of the second kind is recurrent to the unit tensor in the adjoint Riemannian space, the components of the tensor*

$$ "\overset{m}{R}_{irkj} + \frac{1}{2-n}("\overset{m}{R}_{rk}g_{ij} + "\overset{m}{R}_{ij}g_{rk} - "\overset{m}{R}_{rj}g_{ik} - "\overset{m}{R}_{ik}g_{rj}) + $$

(1.5)

$$ + \frac{"\overset{m}{R}}{(n-1)(n-2)}(g_{ij}g_{rk} - g_{ik}g_{rj}) $$

depend only on metrics. Furthermore, they are equal to the components of the conformal curvatore tensor in the adjoint Riemannian space.

Also, we get immediately

COROLLARY 1. *If the adjoint Riemannian space of the Weyl-Otsuki space of the second kind, with tensor* P *recurrent to the unit tensor, is conformally flat, the covariant components of the curvature tensor* $"\overset{m}{R}$ *can be expressed in the form*

$$ "\overset{m}{R}_{irkj} = \frac{1}{2-n}("\overset{m}{R}_{rj}g_{ik} + "\overset{m}{R}_{ik}g_{rj} - "\overset{m}{R}_{rk}g_{ij} - "\overset{m}{R}_{ji}g_{rk}) + $$

$$ + \frac{R}{(n-1)(n-2)}(g_{ik}g_{rj} - g_{ij}g_{rk}) $$

2. Now, I shall consider a case of a non-metric connection, with $m_{ij} = P_{ij}$ and where $g_{ij}, k = \gamma_k m_{ij}$, according to the given regular general connection. In such a case, we shall put

$$ "\Gamma^i_{jk} = "\overset{m}{\Gamma}{}^i_{jk} + H^i_{jk} $$

and use the formula

$$ "R^i_{rkj} = "\overset{m}{R}{}^i_{rkj} + "\overset{m}{\nabla}_k H^i_{rj} - "\overset{m}{\nabla}_j H^i_{rk} + H^s_{rj}H^i_{sk} - $$

(2.1)

$$ - H^s_{rk}H^i_{sj} + "\overset{m}{T}{}^s_{jk}H^i_{rs} $$

We shall consider the fact that connection $"\overset{m}{\Gamma}$ is semi-symmetric, that $"\overset{m}{\nabla}$ denotes the classical covariant differentiation in regard to $"\overset{m}{\Gamma}$.

Taking into account all of these facts, we can actually write (2.1) in the form

$$\text{(2.2)}\qquad {}^{"}R^{i}_{rkj} = {}^{"}\overset{m}{R}{}^{i}_{rkj} + {}^{"}\overset{m}{\nabla}_k H^{i}_{rj} - {}^{"}\overset{m}{\nabla}_j H^{i}_{rk} + H^{s}_{rj}H^{i}_{sk} - H^{s}_{rk}H^{i}_{sj} + \tilde{\pi}_j H^{i}_{rk} - \tilde{\pi}_k H^{i}_{rj}$$

where the overbars mean that the components of this vector are obtained from the original ones, after applying transformation Q.

Using the fact that the right-hand side of (2.2) contains the right-hand side of formula (2.2a) from paper [4] (this one contains two last members more), after interchanging indices $i \leftrightarrow r$ and $j \leftrightarrow k$, addition and all the calculations in almost same manner, we get a similar formula:

$$\begin{aligned}
{}^{"}R_{irkj} + {}^{"}R_{rijk} &= {}^{"}\overset{m}{R}_{irkj} + {}^{"}\overset{m}{R}_{rijk} \\
&+ g_{ik}({}^{"}\overset{m}{\nabla}_j\tilde{\gamma}_r + \tfrac{1}{2}\tilde{\gamma}_r\tilde{\gamma}_j - \tfrac{1}{4}g_{jr}\tilde{\gamma}_p\tilde{\gamma}^p - \tilde{\pi}_j\tilde{\gamma}_r) \\
\text{(2.3)}\qquad &- g_{ij}({}^{"}\overset{m}{\nabla}_k\tilde{\gamma}_r + \tfrac{1}{2}\tilde{\gamma}_r\tilde{\gamma}_k - \tfrac{1}{4}g_{rk}\tilde{\gamma}_p\tilde{\gamma}^p - \tilde{\pi}_k\tilde{\gamma}_r) \\
&+ g_{rj}({}^{"}\overset{m}{\nabla}_k\tilde{\gamma}_i + \tfrac{1}{2}\tilde{\gamma}_i\tilde{\gamma}_k - \tfrac{1}{4}g_{ik}\tilde{\gamma}_p\tilde{\gamma}^p - \tilde{\pi}_k\tilde{\gamma}_i) \\
&- g_{rk}({}^{"}\overset{m}{\nabla}_j\tilde{\gamma}_i + \tfrac{1}{2}\tilde{\gamma}_i\tilde{\gamma}_j - \tfrac{1}{4}g_{ij}\tilde{\gamma}_p\tilde{\gamma}^p - \tilde{\pi}_j\tilde{\gamma}_i)
\end{aligned}$$

We shall denote ${}^{"}\overset{m}{\nabla}_j\tilde{\gamma}_i + \frac{1}{2}\tilde{\gamma}_i\tilde{\gamma}_j - \frac{1}{4}g_{ij}\tilde{\gamma}_p\tilde{\gamma}^p - \tilde{\pi}_j\tilde{\gamma}_i$ by Ψ_{ji}. It can be immediately noted that formula (2.3) is built on the same scheme as formula (1.2). This is the reason to use the same approach to this formula.

After all the calculations, we get the final formula:

$$\begin{aligned}
&{}^{"}R_{irkj} + {}^{"}R_{rijk} + \frac{1}{2-n}(g_{ij}{}^{"}R_{rk} - g_{ik}{}^{"}R_{rj} + g_{rk}{}^{"}R_{ij} - g_{rj}{}^{"}R_{ik}) + \\
&+ \frac{{}^{"}R}{(n-1)(n-2)}(g_{ij}g_{rk} - g_{ik}g_{rj}) =
\end{aligned}$$

$$= {}^{"}\overset{m}{R}{}_{irkj} + {}^{"}\overset{m}{R}{}_{rijk} + \frac{1}{2-n}(g_{ij}{}^{"}\overset{m}{R}{}_{rk} - g_{ik}{}^{"}\overset{m}{R}{}_{rj} + g_{rk}{}^{"}\overset{m}{R}{}_{ij} - g_{rj}{}^{"}\overset{m}{R}{}_{ik}) +$$

$$+ \frac{{}^{"}\overset{m}{R}}{(n-1)(n-2)}(g_{ij}g_{rk} - g_{ik}g_{rj})$$

where ${}^{"}R_{ij}$ and ${}^{"}\overset{m}{R}{}_{ij}$ denote $({}^{"}R_{rijk} + {}^{"}R_{irkj})g^{rk}$ and $({}^{"}\overset{m}{R}{}_{rijk} + {}^{"}\overset{m}{R}{}_{irkj})g^{rk}$ respectively, and ${}^{"}R$, ${}^{"}\overset{m}{R}$ denote ${}^{"}R_{ij}g^{ij}$, ${}^{"}\overset{m}{R}{}_{ij}g^{ij}$ respectively. Now, we can formulate

THEOREM 2.1. *If tensor* P *is recurrent to the unit tensor, according to the connection in the adjoint Riemannian space, and* $m_{ij} = P_{ij}$, *in the Weyl-Otsuki space of the second kind, the tensor on the left-hand side of (2.1) does not depend on the recurrency vector.*

PROPOSITION 3. *Theorem (1.1) from paper [6] is valid for the Weyl-Otsuki space of the second kind and order 1 (see paper [6] for a slightly different terminology), even if vector* π *is not gradient vector field.*

In the general case, we cannot compare the results of the Theorem (1.1) and (2.1) because they carry out different meaning. But, if the curvature tensor has the same features as adjoint Riemann-Christoffel tensor in view of symmetricity and skew-symmetricity for interchanging the places of indices, its Ricci tensor (1.1) will be symmetric, and ${}^{"}\overset{m}{R}{}_{ij}$ in Theorem (2.1) will denote simply a doubled ${}^{"}\overset{m}{R}{}_{ij}$ from Theorem (1.1). This happens if and only if $\tilde{\pi}$ (not π) is a gradient vector.

In this way, combining Theorems (1.1) and (2.1), we get

COROLLARY 2. *If* $\tilde{\pi}$ *is a gradient vector field, the tensor from Theorem (2.1) is equal to the doubled conformal curvature tensor of the adjoint Riemannian space.*

COROLLARY 3. *If* $\tilde{\pi}$ *is a gradient vector field and if the adjoint Riemannian space is conformally flat, the tensor from Theorem (2.1) is identical to zero.*

After some calculation for this condition, we can easily get

COROLLARY 4. *If the condition*

$$\pi_a\left(\frac{\partial Q_j^a}{\partial^k} - \frac{\partial Q_k^a}{\partial^j}\right) + Q_j^a \frac{\partial \pi_a}{\partial^k} - Q_k^a \frac{\partial \pi_a}{\partial^j} \tag{2.5}$$

is satisfied, the tensor from Theorem (2.1) is equal to a doubled conformal curvature tensor of the adjoint Riemannian space. If the adjoint Riemannian space happens to be conformally flat, the mentioned tensor, under condition (2.5) is equal to zero.

REFERENCES

[1] A. Moór, *Otsukische Übertragung mit rekurrentem Maastensor, Acta Sci. Math. Szeged, 1978.*

[2] A. Moór, *Über die Veränderung der Länge der Vektoren in Weyl-Otsukischen Räumen, Acta Sci. Math. Szeged, Vol. 41(1979), 173-185.*

[3] Dj. Nadj, *Ispitivanje tenzora krivina i krivina Weyl--Otsukijevih prostora, Magistarski rad, Novi Sad, 1978.*

[4] M. Prvanović, *Weyl-Otsuki Spaces of the Second and Third Kind, Zbornik radova Prirodnomatematičkog fakulteta u Novom Sadu, 11(1981), 219-226.*

[5] M. Prvanović, *On a Special Connection in a Otsuki Space, Tensor, Vol. 37(1982), 237-243.*

[6] N. Pušić, *Svežnjevi i koneksije na mnogostrukostima, Magistarski rad, Novi Sad, 1982.*

[7] N. Pušić, *Weyl-Otsuki spaces of the second kind with a special tensor P, Zbornik radova Prirodnomatematičkog fakulteta u Novom Sadu, 12(1982), 379-386.*

Received by the editors September 24, 1984.

REZIME

JEDNA NOVA VRSTA OTSUKIJEVIH PROSTORA
SA REGULARNOM METRIČKOM KONEKSIJOM

Rad je uopštenje prethodno razmatranog slučaja Vejl--Otsukijevog prostora druge vrste; neke osobine tenzora krivine se mogu dobiti i uz manje pretpostavki. Iznesene su dve teoreme koje, u neku ruku, predstavljaju generalizaciju teorema iz rada [7], pri čemu vektor rekurencije za tenzor P u pridruženom Rimanovom prostoru ne mora biti gradijent nenula i najmanje C^3 funkcije, već ma kakav vektor.