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A COINCIDENCE THEOREM FOR MULTIVALUED MAPPINGS IN BANACH SPACES

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ABSTRACT

M.A. Krasnoselskii [4] proved that if K is a nonempty closed bounded subset of a Banach space and A, B are operators on K such that A is completely continuous, B is a contraction and $Au + Bv \in K$ for all $u, v \in K$, then the equation $x = Ax + Bx$ has a solution in K. Many papers related to this result have been published. In particular, Melvin [5] has given conditions under which there exists a solution of the equation $x = G(x,Qx)$. We present a generalization of Melvin's theorem for the relation $Tx \in F(x,Q(Tx))$ with F taking values in the family of nonempty closed convex bounded subsets of a Banach space. An application of our result to the theory of differential relations is also given.

1. INTRODUCTION

W.R. Melvin [5] has proved the following theorem:

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Let E *be a Banach space,* K *a nonempty closed convex bounded subset of* E. *Suppose that we have a continuous operator* Q *which maps* K *into a compact subset of* E, *and an operator* G *from* $K \times \overline{Q[K]}$ *into* K *such that* $G(\cdot,y)$ *is continuous for each fixed* $y \in \overline{Q[K]}$ *and* $G(x,\cdot)$ *is a contraction uniformly with respect to* $x \in K$. *Then the equation* $x = G(x,Qx)$ *has a solution in* K.

This result extends the well-known fixed point theorem of Krasnoselskii [4] which combines both the Banach contraction principle and the Schauder fixed point theorem. The purpose of this note is to give some generalization of Melvin's result as a coincidence theorem for multivalued mappings. More precisely, we shall consider the realtion

$$Tx \in F(x,Q(Tx))$$

with F taking values in the family of nonempty closed convex bounded subsets of a Banach space. An application to the theory of differential relations is also given.

2. PRELIMINARIES

Let us denote: by $CB(M)$ - the family of all nonempty closed bounded subsets of a metric space M; by $CCB(M)$ - the family of all nonempty closed convex bounded subsets of a linear normed space M; by $C(M_1,M_2)$ - the metric space of all continuous bounded functions from a metric space M_1 to a metric space M_2, endowed with the usual supremum metric σ.

The sets $CB(M)$ and $CCB(M)$ will be regarded as metric spaces with the Hausdorff metric $Dist_M$, i.e.

$$Dist_M(A,B) = \max[\sup_{x \in A} d(x,B), \sup_{x \in B} d(x,A)] \;;$$

here the distance between any point $x \in M$ and the nonempty subset Z of M is denoted by $d(x,Z)$ $(= \inf\{\rho(x,z): z \in Z\})$.

The Lemma below is an immediate adaptation of the corresponding result of Michael ([6], Lemma 7.1) and is basic in the proof of our main result.

LEMMA. *Let* M *be a metric space and* E *a Banach space. Assume that* $\mu > 1$, $h \in C(M,E)$ *and* $H : M \to CCB(E)$ *is continuous. Then there exists* $h_o \in C(M,E)$ *satisfying*

$$h_o(x) \in H(x), \quad \|h(x) - h_o(x)\| \le \mu \cdot d(h(x),H(x))$$

for each x *in* M.

We shall also use the following fixed point theorem due to Nadler [7]:

Let M *be a complete metric space with the metric* ρ. *Let* $H : M \to CB(M)$ *be such that* $Dist_M(Hx_1,Hx_2) \le \lambda \cdot \rho(x_1,x_2)$ *for* $x_1,x_2 \in M$ *and with a constant* $\lambda < 1$. *Then the multivalued mapping* H *has a fixed point in* M.

3. MAIN RESULT

The result reads as follows.

THEOREM. *Let* M *be a compact metric space,* E *a Banach space with the norm* $\|\cdot\|$, X *a nonempty subset of* E, *and* K *a nonempty closed convex bounded subset of* E. *Let* $Q : K \to M$ *be a continuous mapping,* $T : X \to E$ *a homeomorphism such that* $T[X] \subset K$ *and* $\{T \circ f : f \in C(M,E)\}$ *is a closed subset of* $C(M,E)$. *Suppose that* F *is a mapping from* $X \times M$ *to* $CCB(E)$ *satisfying the following conditions:*

(i) $F[X \times M] \subset T[X]$

(ii) $F(x,\cdot)$ *is continuous on* M *for each fixed* $x \in X$;

(iii) $Dist_E(F(x_1,y),F(x_2,y)) \le k\|Tx_1 - Tx_2\|$ *for all* x_1,x_2 *in* X, $y \in M$ *and with a positive constant* $k < 1$.

Under these assumptions there exists a point x_o *in* X *such that* $Tx_o \in F(x_o,Q(Tx_o))$.

P r o o f. Put $A = C(M,X)$. Define mappings I and Ω as follows: for $f \in A$,

$$I(f) = T \circ f$$

and

$$\Omega(f) = \{g \in C(M,E) : g(x) \in F(f(x),x) \quad \text{for} \quad x \in M\}.$$

Then $I : A \to C(M,E)$, $I[A]$ is closed, and since T is a homeomorphism, it follows that $\Omega[A] \subset I[A]$. It can be easily seen that $\Omega(f)$ is nonempty by Michael [6] (see [3], Theorem B.14) and $\Omega(f)$ is a closed bounded subset of $C(M,E)$; therefore $\Omega : A \to CB(C(M,E))$.

Denote by Φ a choice function for the family $\{I^{-1}(g) : g \in I[A]\}$; here $I^{-1}(g)$ stands for the inverse image of g under I. Let us put

$$G(g) = \Omega(\Phi(I^{-1}(g)))$$

for $g \in I[A]$. Evidently, $G : I[A] \to CB(I[A])$.

Choose a number k_o with $k < k_o < 1$. Suppose that $g_1, g_2 \in I[A]$. Let $g \in G(g_1)$. Since the mapping $x \mapsto F(\Phi(I^{-1}(g_2))(x),x)$ is continuous, by the Lemma, there exists $h_g \in G(g_2)$ such that

$$\begin{aligned}
\|h_g(x) - g(x)\| &\le k^{-1}k_o \cdot d(g(x), F(\Phi(I^{-1}(g_2))(x),x) \le \\
&\le k^{-1}k_o \cdot \mathrm{Dist}_E(F(\Phi(I^{-1}(g_1))(x),x), F(\Phi(I^{-1}(g_2))(x),x)) \le \\
&\le k_o\|T(\Phi(I^{-1}(g_1))(x)) - T(\Phi(I^{-1}(g_2))(x))\| = \\
&= k_o\|(I(\Phi(I^{-1}(g_1))))(x) - (I(\Phi(I^{-1}(g_2))))(x)\| = \\
&= k_o\|g_1(x) - g_2(x)\|
\end{aligned}$$

for $x \in M$; hence $\sigma(h_g, g) \le k_o \cdot \sigma(g_1, g_2)$. Arguing again as above, it follows that if $g \in G(g_2)$ then there exists $h_g \in G(g_1)$ and $\sigma(h_g, g) \le k_o \cdot \sigma(g_1, g_2)$. Thus

$$\mathrm{Dist}_{I[A]}(G(g_1), G(g_2)) \le k_o \cdot \sigma(g_1, g_2) \; .$$

Now, if we apply Nadler's contraction principle given in Sec.2 we conclude that there is g_o in $G(g_o)$. Let $f_o = \Phi(I^{-1}(g_o))$. Then

$$I(f_o) = g_o \in G(g_o) = \Omega(\Phi(I^{-1}(g_o))) = \Omega(f_o),$$

and consequently

$$T(f_o(x)) \in F(f_o(x),x)$$

for each x in M.

Since f_o is continuous, $\{T(f_o(u)) : u \in \overline{Q[K]}\}$ is compact. Therefore, by Schauder's theorem, $y \mapsto T(f_o(Qy))$ has a fixed point, say y_o, in K. Hence $x_o = f_o(Qy_o) \in X$ and

$$Tx_o \in F(f_o(Qy_o),Qy_o) = F(f_o(Qy_o),Q(T(f_o(Qy_o)))) =$$

$$= F(x_o,Q(Tx_o)) \ ,$$

which completes the proof.

4. MULTIVALUED SYSTEMS

Let M be a metric space, E a Banach space, and K a nonempty closed convex subset of E. Consider the multivalued system

$$(+) \qquad \begin{cases} x \in F(x,y) \\ y \in G(x,y) \end{cases}$$

where F,G are two mappings from $K \times M$ into, respectively, 2^K and 2^M (2^X denotes the collection of all nonempty subsets of X). Throughout this part, F is a closed mapping (i.e., $x_n \to x$, $y_n \to y$, $z_n \in F(x_n,y_n)$ for $n \geq 1$ and $z_n \to z$ implies that $z \in F(x,y)$) with convex values and $F[K \times M]$ is conditionally compact in E.

Let us prove first: *If* (1) M *is bounded closed subset of any Banach space* B, (2) $G(x,y) \in CCB(M)$ *for* $(x,y) \in K \times M$, (3) $x \mapsto G(x,y)$ *is continuous on* K *for each* $y \in M$, *and* (4) $\mathrm{Dist}_M(G(x,y_1),G(x,y_2)) \le k\|y_1 - y_2\|$ *for* $x \in K$ *and* $y_1,y_2 \in M$ *and with a positive constant* $k < 1$, *then* (+) *has a solution in* $K \times M$.

Indeed, let $g_o \in C(K,M)$ and $k_o \in (0,1)$. By the Lemma, there exists $g_n \in C(K,M)$ $(n = 1,2,\ldots)$ such that

$$g_n(x) \in G(x,g_{n-1}(x)) \quad \text{and}$$

$$\|g_n(x) - g_{n-1}(x)\| \le k^{-1}k_o \cdot d(g_{n-1}(x),G(x,g_{n-1}(x)))$$

for x in K. Hence we can write

$$\|g_n(x) - g_{n-1}(x)\| \le k^{-1}k_o \cdot \mathrm{Dist}_M(G(x,g_{n-2}(x)),G(x,g_{n-1}(x))) \le$$

$$\le k_o\|g_{n-2}(x) - g_{n-1}(x)\| \le \ldots \le k_o^{n-1}\|g_o(x) - g_1(x)\| \le$$

$$\le k_o^{n-1} \cdot \sigma(g_o,g_1)$$

and therefore the sequence (g_n) is uniformly convergent on K.

Let $f(x) = \lim_{n\to\infty} g_n(x)$ uniformly on K. For $x \in K$, we have

$$d(f(x),G(x,f(x))) \le \|f(x) - g_n(x)\| + d(g_n(x),G(x,f(x))) \le$$

$$\le \|f(x) - g_n(x)\| + \mathrm{Dist}_M(G(x,g_{n-1}(x)),G(x,f(x))) \le$$

$$\le \|f(x) - g_n(x)\| + k\|g_{n-1}(x) - f(x)\|$$

and since $G(x,f(x))$ is closed, it follows that $f(x) \in G(x,f(x))$.

Define: $\Omega(x) = F(x,f(x))$ for $x \in K$. It is easy to see that $\Omega : K \to 2^K$ is a closed mapping with convex values in a compact subset of K. Therefore, by the fixed point theorem of Bohnenblust and Karlin [2], Ω has a fixed point in K*). Let $x_o \in \Omega(x_o)$ and $y_o = f(x_o)$. Then

*) *There is a more general fixed point theorem using condensing mappings. This suggests more general assumption on* F *and* G *giving the existence for* (+).

$$x_o \in F(x_o,y_o), \qquad y_o \in G(x_o,y_o)$$

and our proof is completed.

This proof suggests the following statement: *Let* M *be complete metric space and* $\phi : C(K,M) \to [0,\infty)$. *Suppose that* $y \to G(x,y)$ *is a closed mapping on* M *for each* $x \in K$. *If for every* $g \in C(K,M)$ *there exists* $h_g \in C(K,M)$ *such that* $h_g(x) \in G(x,g(x))$ *on* K *and* $\sigma(g,h_g) \leq \phi(g) - \phi(h_g)$, *then* (+) *has a solution.*

From the above as a corollary we obtain our first result about (+) when B is a uniformly convex Banach space.

As a matter of fact, let us assume that the conditions (1) - (4) are satisfied, and in addition, B is a uniformly convex space. Let $g \in C(K,M)$. By the immediate adaptation of Lemma 5.2 of Banks and Jacobs [1], there exists a uniquely determined sequence (g_n) of $C(K,M)$ such that

$$g_n(x) \in G(x,g_{n-1}(x)) \quad \text{and}$$

$$\| g_n(x) - g_{n-1}(x)\| = d(g_{n-1}(x),G(x,g_{n-1}(x)))$$

for $x \in K$, where $g_o = g$. Hence

$$\sum_{n\geq 1} \sigma(g_n,g_{n-1}) \leq \sigma(g_1,g_o) \sum_{n\geq 1} k^{n-1} < \infty .$$

Putting

$$\phi(g) = \sum_{n\geq 1} \sigma(g_n,g_{n-1})$$

we shall have $\sigma(g,g_1) = \phi(g) - \phi(g_1)$ and we have finished.

Finally, let us remark that similar results can be obtained also as coincidence theorems.

5. APPLICATION

Let $J = [0,1]$ and let D be the set of all $x \in \mathbb{R}^n$ (the n-dimensional Euclidean space with the zero element θ) such that $|x| \leq C$. We shall consider a differential relation

$$u'(t) \in U(t,u(t),u(t)), \qquad u(0) = \theta$$

where U is a given continuous mapping of $J \times D \times D$ into $CCB(\mathbb{R}^n)$.

Assume in addition that

$$\mathrm{Dist}_{\mathbb{R}^n}(U(t,x,y),\{\theta\}) \leq C$$

and

$$\mathrm{Dist}_{\mathbb{R}^n}(U(t,x_1,y),U(t,x_2,y)) \leq L|x_1 - x_2|$$

for $t \in J$ and x, x_1, x_2, y in D.

Let $r > \max(1,2L)$. We put:

$$X = \{w \in C(J,\mathbb{R}^n) : |w(t)| \leq C \quad \text{for } t \in J\},$$
$$K = \{w \in C(J,\mathbb{R}^n) : |w(t)| \leq C\cdot\exp(-rt) \quad \text{for } t \in J\}.$$

Define mappings T and Q by

$$(Tw)(t) = \exp(-rt)w(t),$$
$$(Qw)(t) = \int_0^t \exp(rs)w(s)ds$$

for $w \in C(J,\mathbb{R}^n)$. Moreover, for $u \in X$ and $v \in \overline{Q[K]}$ (the closure of $Q[K]$ in $C(J,\mathbb{R}^n)$), we denote by $F(u,v)$ the set of all functions $t \mapsto \exp(-rt)f(t)$ such that $f \in C(J,\mathbb{R}^n)$ and $f(t) \in U(t, \int_0^t u(s)ds, v(t))$ on J.

The set $E = C(J,\mathbb{R}^n)$ will be considered as a Banach space with the supremum norm $\|\cdot\|$. It is easy to see that K is a closed convex bounded subset of E, $T[X] \subset K$, $M = \overline{Q[K]}$ is compact, and $F[X \times M] \subset T[X]$. The closed convex bounded set $F(u,v)$ is nonempty by Michael [6], and therefore F is a mapping from $X \times M$ to $CCB(E)$.

Suppose that u_1, $u_2 \in X$ and $v \in M$. Let $f_1 \in F(u_1,v)$. Let $f_1(t) = \exp(-rt)z_1(t)$ with $z_1 \in E$ and $z_1(t) \in U(t, \int_0^t u_1(s)ds, v(t))$ for $t \in J$. By the Lemma, there is $z_2 \in E$ such that $z_2(t) \in U(t, \int_0^t u_2(s)ds, v(t))$ and

$$|z_1(t) - z_2(t)| \leq 2 \cdot d(z_1(t), U(t, \int_0^t u_2(s)ds, v(t))) \leq$$

$$\leq 2 \cdot \mathrm{Dist}_{\mathbb{R}^n}(U(t, \int_0^t u_1(s)ds, v(t)), U(t, \int_0^t u_2(s)ds, v(t))) \leq$$

$$\leq 2L \int_0^t |u_1(s) - u_2(s)|ds =$$

$$= 2L \int_0^t \exp(rs)|(Tu_1)(s) - (Tu_2)(s)|ds \leq$$

$$\leq 2L |||Tu_1 - Tu_2||| \int_0^t \exp(rs)ds < 2L \cdot r^{-1} \cdot \exp(rt) |||Tu_1 - Tu_2|||$$

on J. Put $f_2(t) = \exp(-rt)z_2(t)$ for $t \in J$. Then $f_2 \in F(u_2, v)$ and

$$\sigma(f_1, f_2) = \sup_{t \in J} \exp(-rt)|z_1(t) - z_2(t)| \leq 2L \cdot r^{-1} |||Tu_1 - Tu_2||| .$$

This implies

$$\mathrm{Dist}_E(F(u_1, v), F(u_2, v)) \leq \frac{2L}{r} |||Tu_1 - Tu_2|||$$

for all u_1, $u_2 \in X$ and $v \in M$. Modifying the above reasoning, we obtain that $v \mapsto F(u,v)$ is a continuous mapping from M to CCB(E) for every $u \in X$.

Now, according to the Theorem applied to $E = C(J, \mathbb{R}^n)$ and our X, K, T, Q, F and $M = \overline{Q[K]}$, there exists $u_o \in X$ such that $Tu_o \in F(u_o, Q(Tu_o))$; therefore

$$u_o(t) \in U(t, \int_0^t u_o(s)ds, \int_0^t u_o(s)ds$$

for $t \in J$, and we are done.

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REZIME

TEOREMA KOINCIDENCIJE ZA VIŠEZNAČNA PRESLIKAVANJA U BANAHOVOM PROSTORU

U ovom radu dokazano je jedno uopštenje teoreme Melvina [5] za relaciju $Tx \in F(x,Q(Tx))$ gde F uzima vrednosti u familiji nepraznih zatvorenih ograničenih podskupova Banahovog prostora. Data je primena dobijenih rezultata na diferencijalne relacije.